A MATHEMATICAL MODEL FOR POLITICAL DISTRICTING WITH COMPACTNESS CONSIDERATION AND AN APPLICATION TO KENTUCKY SENATE DISTRICTING

BY

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THESIS

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ABSTRACT

The basic redistricting problem is defined as aggregating a set of base (indivisible) units into contiguous geographical areas with almost equal voter population, called districts, and districts into a district plan. The method used to create districts can have a substantial impact on election results and therefore the laws passed. Districting is a very difficult problem that usually leads to severe political debate no matter who does it and what the result looks like. The problem becomes even more convoluted when the very people that can benefit most from generating district plans are given control. Traditionally this problem is solved by map-making skills based on subjective criteria (typically partisan interests). In addition to the two basic criteria, namely contiguity and population equality, other spatial and socioeconomic criteria have been considered when generating district maps. These additional criteria may involve compactness, community integrity, racial concerns, etc. Generating district maps by simultaneous consideration of various criteria leads to a challenging combinatorial problem; therefore, sophisticated methods are needed for this purpose. Development in computer technologies in recent decades made it possible to draw district boundaries using computer software although this approach has not been adopted in most states.

This thesis introduces a mathematical programming model, specifically a linear mixed-integer program, to create a politically unbiased districting plan that can be augmented to meet the needs of the user. This method is empirically tested in an attempt to create a state Senate district plan for Kentucky. Kentucky was chosen in particular because of a Supreme Court case that eventually held that the proposed districting plan violated the State Constitution and a new plan needed to be created that contained compact and contiguous districts each consisting of a population within 5% of the ideal all the while dividing counties as rarely as possible. In

addition, both the number of senate districts and base units in the State of Kentucky are relatively small, making the state a good choice for model testing.

The thesis concludes with an assessment of the relationship between compactness and the demographic composition of the district maps generated by the model vis-à-vis the actual state Senate district map.

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Chapter 1: Introduction

1.1 Motivation and Purpose

This chapter gives an overview of the political redistricting process in general and states the goals of this thesis.

In the United States, states are divided into political districts for election purposes. Each district elects one representative. Every ten years, each state is required to update the state Senate and state House of Representative districts in accordance with the state and federal laws and the results of the newest United States Decennial Census. The United States Constitution also requires congressional seats to be reapportioned among the states after each decennial census. Any state with more than one district must adjust its district lines so each House of Representatives district contains equal numbers of people as much as possible.

All states require that districts are contiguous and preserve population equity to the extent possible. A contiguous district is a district where a person can start at any place within the district and travel to any other location in the district without crossing another district. Population equity requires districts to have approximately equal populations. This is in accordance with the "One Person, One Vote" democratic principle. The purpose is to insure that the voting power of everyone in the state is equal. Since every district elects one representative, violation of this rule would give the constituents of a district with smaller population more voting power than the constituents of a largely populated district. A fictitious example gives weight to this issue: Suppose district A has one hundred thousand people, while district B has one million. The votes from district A would then carry ten times the voting power of the votes from district B. The actual specification of how far district populations can vary from the average population is left to the states, but it is generally five to ten percent above or below the average district population for state legislative districts and can be as fine as one percent or less for congressional districts.

Some states set additional redistricting criteria. Districts in Illinois, for instance, must be 'compact'. The definition of compactness and the level of compactness that is acceptable are not given specifically. Some states require community integrity, which refers to minimizing the number of communities that are divided between two or more districts. Communities can denote counties, cities, other municipalities, or areas with business relationships. Still other states require proportional representation of minorities. If a minority group comprised 30% of the state population and was proportionately distributed throughout the state, they would be unlikely to have enough votes in any district to receive minority representation. Proportional representation of minorities aims to create a redistricting plan where, in this example, 30% of the districts have a majority of the residents from the minority group. The 'majority' may sometimes mean more than an absolute 50% majority and can be specified as high as 60% because of assumed or known minority reluctance to vote. This increases the likelihood that the minority group will be represented in proportion to their state levels. A thorough review of the common redistricting criteria will be covered in Chapter 2.

Redistricting in all but twelve states is the responsibility of the state legislature. The other twelve states organize a committee to create the redistricting plans. In certain states, once redistricting plans are created, they must be submitted to the U.S. Department of Justice for approval. If the Department of Justice does not accept the plan, the prior plan may continue or a new plan may be submitted for approval.

1.2 Why Kentucky?

Occasionally, redistricting plans are disputed by opposing political parties or residents and taken to court. The particular motivation for this thesis is a Supreme Court case, *Fisher v. State Board of Elections* (1994). In this case, a Kentucky resident claimed the proposed redistricting plan violated Section 33 of the State Constitution, which states,

> *"The . . . General Assembly . . . shall divide the state into thirty eight Senatorial Districts and one hundred Representative Districts, as nearly equal in population as may be without dividing any county, except where a county may include more than one district . . . ."*

The plan divided 48 out of 120 counties in the House of Representatives redistricting plan and 19 counties in the Senate redistricting plan. The Supreme Court ruled in favor of the plaintiff concluding that, "as between competing concepts of population equality and county integrity, the latter is of at least equal importance." The Supreme Court interpreted that Section 33 requires a redistricting plan to divide the smallest number of counties possible while preserving a maximum district population deviation from the average of 5% above or below. The minimum number of county divisions achievable while meeting the population equity requirement was unknown. This is the main motivation of the modeling approach introduced in this thesis. Here I address the Kentucky Senate districting as a case study, but with proper modifications, the same approach can be used for generating state representative districts or congressional districts as well.

In addition to the court case mentioned above, the relatively small size of the state, in terms of the number of legislative districts, was another reason for choosing Kentucky. As will be explained later in the model section, the districting problem is computationally complex especially when using a mathematical programming model for this purpose. The number of districts to configure and the number of base units to be aggregated when districting are crucial for computational complexity. Increasing the number of base units and/or the larger the number of districts to configure, increases the complexity of the model. In order to test the solution difficulty/convenience of the model developed in this thesis, a small state like Kentucky is a good choice.

1.3 The Modeling Approach

In the past, many approaches have been proposed to create redistricting plans, some with greater success than others. Maps were originally created by hand using the aid of census data and statisticians. Recently, district maps have been created using heuristic approaches.

1.3-A. Terminology

I define here terms used in this thesis and throughout political redistricting literature.

Unit or Base Unit: Relatively small indivisible areas within a state. They are aggregated to other units to form districts. Redistricting models often use counties, census tracts, or census blocks as the base unit.

District: A collection of units that elects a representative.

District Plan: A set of Districts that encompasses the state or area to be redistricted.

Representative: An elected official that acts on the behalf of his constituents.

In this study, an optimization model, specifically a linear integer program, is introduced for aggregation of base units that form a district. The model is a variation of a generic formulation, known as the *p-center formulation,* adapted to the particular problem addressed here. Similar problems arise in business districting where customers are assigned to a given set of business agents (processing or supplier locations, such as plants or warehouses) where distances between customer and agent locations and workload of individual agents may be taken into account when doing the assignment. Another similar problem occurs in school districting; where school locations and the locations of neighborhoods in which attending students live are important considerations (gender and racial compositions are other important assignment criteria). These types of ‘clustering’ problems involve distances between a number of central locations and the locations assigned to each center (thus the name ‘p-center’) among other considerations and objectives. The model developed here chooses units to be district centers and attaches the closest units to those centers in such a way that the total distance between the district center and the units assigned to that center, summed across all districts, is minimized. This is done by defining yes-no type decision variables (‘yes’ means assign, ‘no’ means do not assign a particular unit to a particular district center) represented by binary (0-1) variables in the model. The model considers a number of potential centers for districts and selects a subset of them that actually serve as district centers together with the unit assignments. The district centers have no ‘real world role’, rather their artificial role is to make the mathematical model work and allow the aggregation of base units according to the specified criteria. The total distance can be viewed as a measure of compactness, but this is actually a secondary purpose. The main purpose of this model formulation (approach) is to create contiguous districts. Minimization of the sum of distances to district centers generally selects units that are adjacent to their centers or to each other, thus forming spatially contiguous clusters of base units. The mathematical representation of the common criteria used in the study will be explained in the model chapter.

1.4 Empirical Application to Kentucky

In Chapter 4, the optimization model briefly outlined above will be applied to answer the question posed by the holding in *Fisher v State Board of Elections* (1994); that is how to create a redistricting plan for the Kentucky Senate with the least division of counties (considered here as communities) and adhering to a strict criterion of 5% allowable deviation from the average district population. The resulting districting plan will be compared to the proposed plan that was the focus of *Fisher* as well as to the current Kentucky Senate District map. The comparisons will focus on reduction in county divisions and compactness.

1.5 Effects of Compactness on Socioeconomic Composition of Districts

Besides the very first political purpose (service), namely the election outcome, spatial configuration of political districts may have substantial implications in terms of socioeconomic characteristics of the districts. Consideration of compactness is of particular interest and may be crucial when grouping people with different political preferences, economic power, ethnic/racial background, and professional occupation since people with similar characteristics are more likely to be located in the same area. Hence, compactness is hypothesized to result in more homogenous district compositions, which is yet to be tested. Having designed a compact district map as an alternative to an existing district map gives us a unique opportunity to test this hypothesis.

Past research regarding compactness of political districts has focused on how there is no perfect measure of compactness (Young, 1988) or whether compactness is really an indication of an absence of ‘gerrymandering’[1] (Altman, 1995). Chapter 5 attempts to determine the relationship between compactness and the grouping of people with similar socioeconomic characteristics. Specifically it aims to answer questions including: Is a compact district more likely to be composed of people working in the same industry? Are compact districts more likely to be comprised of people with similar educational backgrounds? Are compact district more homogenous in terms of racial/ethnic background of the constituents? This comparison is done at

[1] See p.15 for the meaning of this term.

two levels in the present study. The aggregate (plan) level comparison is checked for differences between the actual (current) Kentucky Senate districting map and the districting map created using the model developed in this thesis. The same tests are completed at the district level to determine if within a single plan there is a substantial difference between the compositions of inhabitants of a compact district versus a less compact district configuration.

1.6 Summary and Outline of the Thesis Study

This thesis is organized in six chapters. Following this introduction chapter, Chapter 2 will review the redistricting process in the United States with specific districting rules applicable to the case study considered here. Chapter 3 will focus on development of an optimization model for the general political redistricting problem. Chapter 4 implements the mathematical model to a particular redistricting problem, namely Kentucky State Senate districting, as a test problem. The application is based on the proposed redistricting plan for the Kentucky State Senate that was taken to the Supreme Court, as mentioned above. Chapter 5 attempts to determine what relationship, if any, the compactness criterion has on the grouping of similar people within a district and possible political outcomes. The final chapter summarizes the main contributions of this thesis, strengths and shortcomings of the methods used, and directions for future modeling research using mathematical programming.

Chapter 2: Literature Review

Prior to the use of computers for redistricting in the 1960's, political redistricting was the job of mapmakers and statisticians. This review focuses on common criteria in the redistricting process and methods that have been applied to political redistricting.

2.1 Court Cases

Three Supreme Court cases in the early 1960's led to sweeping changes in political districting. The true landmark case was *Baker v Carr* (1962). In a six-to-two vote, it was decided that redistricting was a matter that could be evaluated by the judicial system. This was in direct contradiction to *Colegrove v Green* (1946) which held that apportionment and redistricting was solely a political matter. *Baker* allows the courts to make decisions on redistricting cases. Two years later, *Baker* was utilized in *Reynolds v Sims* (1964). At the time, the population of a state Senate district in Alabama was 41 times greater than the least populous senate district. The *Reynolds* case held that according to the "one person, one vote" principle, state legislative districts should have approximately equal populations. In a similar case during the same year, the Supreme Court held that U.S. Congressional districts should also have approximately equal populations (*Wesberry v. Sanders*, 1964). The ruling was not applied to U.S. Senate districts.

Following these cases, almost every state needed to be reapportioned at some level. This is the perfect research opportunity as far as academics are concerned, an interesting topic with new Supreme Court precedent, and many unknowns to research. The next three sections of this chapter introduce common criteria that have been implemented in order to reapportion districts, discuss the previous attempts at political redistricting, and review the p-center problem and other mathematical techniques I use to create my political districting model.

2.2 Review of Common Political Districting Criteria

There are many criteria that one can implement when redistricting. *Williams* (2005) organizes them into three broad categories. They are geographic criteria, demographic criteria, and political criteria. Implementation of a selection of these criteria is described in Chapter 3.

2.2-A. Geographic Criteria

Four geographic criteria have been incorporated in redistricting plans. They are contiguity, compactness, convexity, and community integrity. Imposing district contiguity means that it must be possible for a person to start at a point within a district and travel to any other point within that district without crossing into another district. This is a required criterion in all state constitutions and it is usually very easy to determine visually if this criterion is satisfied. Issues arise when a state contains an island or any other disjoint piece, as is the case for the Kentucky Bend.

Compactness is another geographic criterion that is often imposed. The definition of compactness is still debated[2]. In general, a compact district is circular or square in shape. These simple shapes give the impression[3] that the districts were created sans 'gerrymandering' and that more people with similar interests are inhabitants of the same district. Chapter 5 of this thesis shows that compactness is no guarantee of either implication.

A less used criterion related to compactness and contiguity is the creation of convex districts. A district is convex if a line segment connecting any two points within a district is entirely within that district or equivalently if all interior angles of the district are less than 180 degrees. A person must be able to travel in a straight line from any point in a district to any other point in the district without entering another district. This insures that a district does not wrap

[2] Compactness will get an entire chapter devoted to it in this thesis.
[3] Which may or may not be true.

around another district or create a doughnut shaped district. Spann et al. (2007) were able to create convex districts in all cases except where the district bordered a state boundary that was not convex, applying their model to small, medium, and large states. Convexity is similar to compactness in that both imply that the districts were not ‘gerrymandered’.

The final geographic criterion is community integrity. A community can be defined based on preexisting borders as in counties or cities. A community can also be based on economic, educational, or social similarities (Williams, 2005). For example in 2001 the Kansas Legislative Research Department went to town hall meetings and asked residents what is important to consider when redistricting (“Testimony Presented at Town Hall Meetings,” 2001). An overwhelming majority of the feedback showed that residents wanted to be in the same district as residents in the same school district. It was also important to aggregate cities and counties containing many workers commuting to common workplaces. One resident of Reno county stated “…commonality of interests with Salina and Garden City because of agriculture-related manufacturing…” is a reason for these cities to be in “…the first district.” Preserving communities is easy to codify into a redistricting plan, but it can adversely affect other criteria. Preserving communities will likely force districts to be less compact. If a community of interest is densely populated the community integrity criterion can make the model infeasible due to population equity violations.

2.2-B. Demographic Criteria

The next grouping of criteria is demographic criteria. The two criteria imposed are population equity and proportional minority representation.

Population equity among districts, set forth by *Reynolds v. Sims* (1964) and *Wesberry v. Sanders* (1964), requires state legislative and congressional districts to have approximately equal populations. The allowable population deviation varies and has been independently decided in multiple Supreme Court cases. Equally populous districts are supposed to ensure the principle of "One Person, One Vote." However, equally populated districts may not necessarily guarantee the one person-one vote property in an absolute sense since the inhabitants of a district may include people that are not eligible to vote (minors and convicted felons). If the proportion of eligible voters is much less in one district than others, the voters in that district will have more voting power than other voters. Furthermore, inclusion of an 'indivisible base unit' adds a fixed number of people to that district; therefore, due to discrete additions an absolute equality of district populations may not be arithmetically possible. Therefore, in practice the one person-one vote principle is implemented by imposing 'approximately equal' district populations instead of 'exactly equal'.

Incorporating population equity into a redistricting plan can take many forms. The most common measure, known as extreme deviation, sets a limit on the maximum percent above or below the average district population that an individual district may deviate. This sets an allowable range for district populations. Another way to impose population equity is by setting a limit on the mean squared deviation. This incorporates the population of each district instead of just the single extreme district as in the extreme deviation case. The final criterion that is rarely used today is setting a limit on the minimum control percentage. The minimum control percentage is the smallest percent of the total population that could elect a majority of representatives. To find the minimum control percentage, take the percent of the total population within the least populous fifty percent plus one districts (Williams, 2005).

For a districting plan to include proportional minority representation, minority groups must have the opportunity to elect representatives of their choice. It would be ideal to have the percentage of minority representatives and the share of the minority population in the state to be more or less the same (to the extent this is practicable). Hypothesizing that minority groups would prefer to elect a minority candidate, which may not always be the case, a fair way to accomplish this outcome is to draw 'minority district' boundaries so that those districts contain a majority of the minority group. If the above hypothesis is true, this facilitates approximately equaling the share of the minority groups in the total population and the share of the minority representatives.

The use of proportional minority representation began in the mid 1960's following the Voting Rights Act of 1965. The Voting Rights Act contained two important implications for redistricting. Section 2(a) and (b) read:

> *"No voting qualification or prerequisite to voting, or standard, practice, or procedure shall be imposed or applied by any state or political subdivision in a manner which results in the denial or abridgement of the right of any citizen of the United States to vote on account of race or color…"* and this is violated when minority voters *"…have less opportunity than other members of the electorate to participate in the political process and to elect representatives of their choice."*

States that have severely hindered minority groups in the past need preclearance from the U.S. Department of Justice before a redistricting plan could be approved. Preclearance has not been a great deterrence. In the past 10 years, the U.S. Department of Justice's objection rate has been less than 0.1%. In addition, *Bartlett v. Strickland* (2009) ruled that a district plan needed adjustment to preserve minority representation only when a minority group comprises greater than 50% of the voting age population. An issue similar to the problem with population equality is that using the total population of the minority group does not necessarily give minorities an opportunity for "One Person, One Vote." Using the voting age population might be a better option. Even that might not lead to the desired outcome due to the peculiarities in minority voting behavior, namely reluctance to participate in the political process.

A serious compromise required to create minority districts is the loss of compactness. It may not be feasible to create a minority district that is compact or follows community boundaries. An example of this is in the 1992 North Carolina redistricting plan, where a minority district included parts of 25 separate counties. A majority-minority district in Illinois displays the trade-off between compactness and creating majority-minority districts. Two Hispanic communities are attached by a section on the West side of the district that follows Interstate 294 that does not contain any residents. (See Figure 1)

Figure 1: Illinois Congressional District 4[4]

[4] Source: National Atlas of the United States

2.2-C. Political Criteria

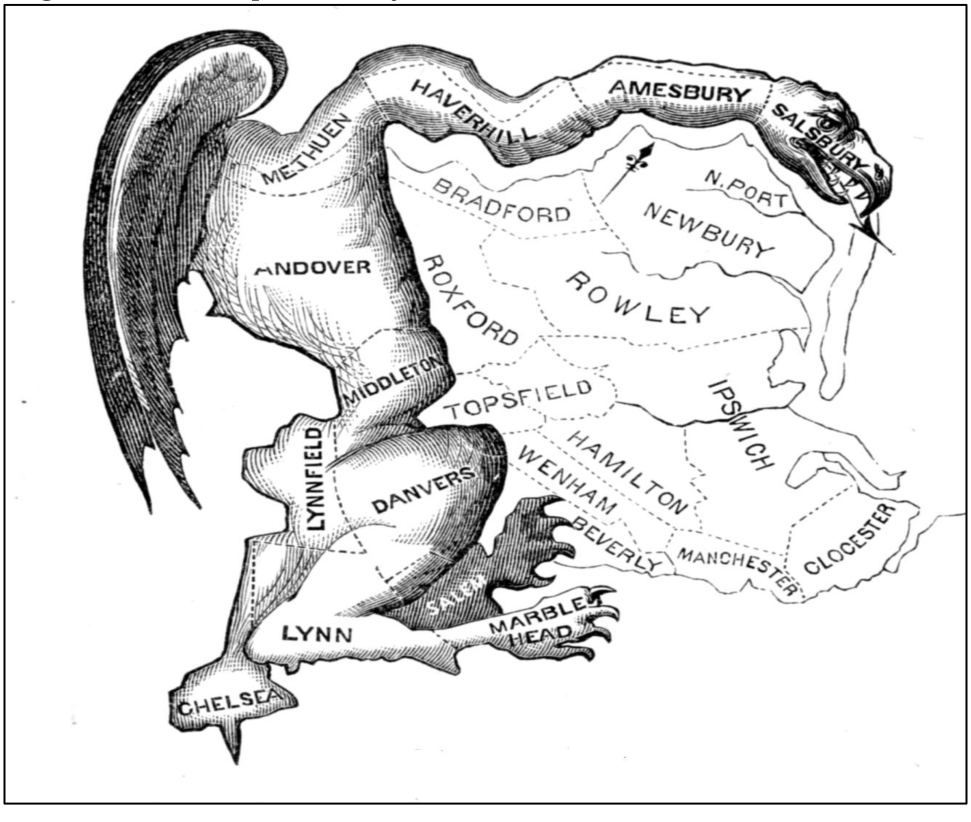
The final category of often-included criteria is of a political nature. Whether it is beneficial to the general population or not, political criteria are included in redistricting plans to encourage the reelection of incumbents and benefit the personal interests of a political party when the legislature is in charge of redistricting. Two criteria fall into this category, voting results equaling election results and deviating from the previous district boundaries as minimally as possible. Some researchers believe it is unethical to include political criteria on a districting plan (Spann et al., 2007, p. 284).

If political criteria are used, it would seem fair to attempt to create districts that increase the likelihood that aggregate voting results are proportional to election results. In other words, the distribution of seats won by political parties reflects the distribution of the votes they received. For instance, if the Republican Party gets 80% of the total statewide votes in an election, the preferable outcome is 80% of the districts to have elected Republican representatives. This may not always happen, as one party may win a few districts with a wide margin but lose many districts with a narrow margin. In the real world, this form of representational equity is not always guaranteed. The first reason is numerical. Since each district elects one representative it may not be possible to get voting results to match the election results (seats won). This can be dealt with by allowing election results to deviate slightly from voting results, but this deviation is to be minimized. A measure of how far election results deviate from voting results is the mean squared deviation. This is calculated by taking the average of the squared deviations from the optimal voting results to each political party's election results. The second obstacle to creating districts where election results are approximately equal to voting results is the time lag between redistricting and actual voting.

When district boundaries are drawn, the future voting results are unavailable and not everyone votes consistently through time for the same political party. The only information available is voting results from the previous election(s), which may or may not accurately predict future voting results. Furthermore, the district configuration (boundaries) may affect the candidates running for election in that district, which in turn may change (and often does) the voting pattern of the same people. Therefore, past election results (revealed voting preferences) for a particular geographical area may not be used as a good predictive measure for recent election if the candidates and district boundaries are different in those elections.

A district plan that intentionally draws district boundaries to subvert election results from equaling voting results is termed as 'gerrymandered'. The term gerrymandering comes from an 1812 newspaper description of a district signed into law by then Massachusetts governor, Elbridge Gerry to preserve the power of his Democratic - Republican Party.

Figure 2: 1812 Map of a Gerrymandered District from the Boston Gazette



The newspaper article claimed that the district resembled a salamander, and combined ‘Gerry’ and ‘salamander’ to coin the term gerrymander. Gerrymandering does not always have a negative connotation. Including minority districts in a redistricting plan is also a form of gerrymandering.

A variation on the election results equal to voting results criterion is to use the same principal to create ‘safe’ and ‘swing’ districts. A safe district is a district where the political party of the representative is unlikely to change. Swing districts in contrast are districts where outcome of elections are unclear and where small changes in voter turnout or candidate popularity can have an influence (Williams, 2005). Creating safe districts may be beneficial to the party likely to win that safe district, but a safe district can negatively affect that party if too many votes are ‘wasted’ in the safe district. Any votes beyond the 50% margin needed to win a district are in effect wasted votes that could have been utilized in a different district alternatively by switching a suitable part of the district with an equally populated part of an adjacent district if those votes might lead to winning that district also. Purposely maximizing the wasted voters of one political party in a district is a type of gerrymandering known as ‘packing’.

Another political criterion implemented in districting plans requires a new districting plan be as similar to the existing plan as possible. This has the simplicity of sticking to the status quo. It is beneficial to incumbents that would otherwise need to campaign to a completely different group of residents if districts were drawn without regard to old district boundaries. It may also be beneficial to voters that would not need to learn about the ideals of completely new candidates.

One potential problem with this criterion is that the original plan is based on ten-year-old information that may be substantially different from the current census data. Adhering to the old plan may worsen population equity if significant demographic changes and migration in or out of that district has occurred. In addition, preserving previous district boundaries would sustain the 'errors' of the past if some of those have been gerrymandered in the past[5]. All of the criteria discussed above have strengths and weaknesses. The inclusion of one criterion may reduce effectiveness of another. For example, enforcing community integrity is likely to reduce compactness, or reducing gerrymandering may result in a loss of minority representation. Not all criteria need to be included in a districting plan. A redistricting committee or legislature tasked with redistricting a state can utilize a selected subset of these criteria to create a district plan that satisfies the needs.

2.3 Review of Methods Used for Political Redistricting

The political districting problem is a complex and methodologically challenging combinatorial problem. Therefore, it has attracted the research interests of a very diverse group of scholars from fields such as political science and geography to operations research and economics. This section will cover some of the major methods applied to political redistricting.

Papayanopolous (1973) groups redistricting methods into four categories, allowing individual methods to bleed into multiple categories. The first grouping is heuristic versus exact. The second is sequential versus simultaneous. The third is enumerative and non-enumerative. The final category is numerical versus graphical. *Williams* (1995) attempts to simplify the categorization of political districting methods by only using one yardstick: heuristic versus exact. I will use a categorization approach that *Williams* alludes to in his conclusions. I use the general process used to redistrict to categorize the methods. The general processes are as follows: 1) Create individual districts in isolation, and then combine districts to create a district plan; 2) Attach base units to district centers to create a district plan; 3) Start with an existing plan and

[5] assuming gerrymandering is unwanted.

modify an existing plan by reallocating base units to different districts; 4) Geographical mapping methods. The conclusion will briefly discuss some recent methods used for political redistricting.

2.3-A. Individual Districts Combined into Plans Methods

A method proposed in 1961 by *Vickrey* intends to create compact, contiguous, and relatively equally populated districts. The method begins with a 'seed' unit chosen near the edge of the state. The closest adjacent unassigned base units are attached to the seed unit one-by-one until the district reaches the population limit. The process is repeated choosing seed units throughout the state until no more districts may be created. The modeler is allowed to swap base units to other districts to correct for population issues (Vickrey, 1961). This method has a very serious problem of leaving unassigned base units or gaps between districts (Williams, 2005). Clearly a different selection of the seed units may lead to a very different redistricting plan, thus the number of alternative plans using this approach is practically infinite. Which direction to proceed while appending adjacent units is another issue. This also leads to too many alternatives to consider. Thus, although being technically simple (but note the deficiency of possible gaps), this approach is not a practical one due to the sheer number of alternatives that may be taken into consideration.

Harris, citing the problems with the method of creating districts one at a time used by *Vickery*, proposed a method that yields a "…simultaneous solution" (Harris, 1964, p. 221). *Harris*' method is two staged. In the first iteration, a square grid is superimposed on top of a state map. The squares were supposed to equal the size of the area that could contain the average district population of the most densely populated area. The squares were then attached to create contiguous rectangular districts each meeting the given population criteria. All of the possible plans meeting these criteria were enumerated. Each plan was evaluated based on compactness, measured as the sum of the length minus width of each district, with less compact districting plans being discarded. The ‘optimal’ plan is eventually reached. Once the best plan is found, the lines could be tweaked to account for community or political interests. This approach is severely limited in the fact that it is not clear how to decide which squares in the grid should be grouped together. Even a reasonably sized plan was not possible using this method since this plan was to be completed by hand.

Thoreson & Liittschwager (1967) saw the value in *Vickery* and *Harris*’ methods. They expanded on each method in an attempt to correct the flaws in each approach. Their approach expanded on *Vickery*’s method to reduce the likelihood of reaching a ‘locally optimal’ solution and correct the gaps or residual units often left when the model was completed. To discourage locally optimal solutions, the algorithm was run many times. Each run selected a different starting seed unit then followed *Vickery*’s method of attaching the closest base units until the population limit was reached. Instead of one solution, the model could choose the best district plan among several. In order to deal with leftover units, the population limit (or ‘quota’ as they called it) was allowed to ‘float’ after each successive district was created. The population quota is the total unassigned population divided by the number of remaining districts to be created (Thoreson & Liittschwager, 1967). Unfortunately, even medium sized state districting problems were not solvable using this method.

Thorseon & Liittschwager then expanded upon *Harris*’ method by including the methods applied to *Vickery*’s method to break free of locally optimal solutions as well as the floating population quota. The main change made to *Harris*’ method was the creation of a structured approach to attaching base units to the seed unit. The closest previously unattached base units would be attached to the seed units in a counterclockwise rotation until the population quota was reached.

Garfinkel and Nemhauser (1970) proposed the first applicable optimization method for political redistricting (Williams, 2005). It was a two-stage model that first determined all the possible individual districts that could be made that satisfied the three given criteria of population equity, contiguity, and compactness. Following the completion of stage one, known as 'column generation' in optimization literature, stage two would find the optimal combination of districts that minimized population deviation (Garfinkel & Nemhauser, 1970). This method bears the same deficiencies of Vickrey's approach, plus more. First, it is unknown how many individual districts should be generated in the first stage and what kind of systematic procedure would be needed to accomplish this. Second, unless all or a sufficiently large set of possible district configurations have been generated in the first stage, a feasible solution without gaps or unaccounted units (not belonging to any district) may not always be guaranteed.

2.3-B. Allocation of Base Units to District Centers Methods

Weaver and Hess (1963) used a heuristic method to create a set of possible district plans and eventually choose the best plan among the choices. A set of base units was chosen to be district centers; one district center for each district. A compactness measure was minimized subject to the constraint that the population of each district must be equal. Weaver and Hess used population-weighted moment of inertia as the compactness measure. This measure is the sum of population times the distance squared of each base unit to its respective district center. The best solution (district plan) was obtained subject to chosen starting centers. The algorithm was re-run numerous times with a new set of district centers chosen each time. Once a sizable set of district plans was obtained, *Weaver and Hess* iteratively reviewed each plan to select the best plan. Each plan was reviewed and if any districts were not contiguous or if the plan was 'dominated' by the best plan up until that point, the district plan was discarded. If the plan under review contained contiguous districts and improved upon the previous best plan, it was adopted as the best plan and the previous best plan was discarded. The method proposed in this thesis is based on the same idea, however unlike the heuristic method used by *Weaver and Hess* an exact optimization method is used in this study. It is a well-known fact that heuristics in general do not produce exact optimal solutions, in this case ultimate compactness, and in some cases, the deviation from the exact optimum solution (sub-optimality) can be as much as 25%. Without knowing the exact optimum solution there is no way of knowing how sub-optimal the solution is. Therefore, the approach used in this thesis would be a preferred approach (as long as the integer programming model is computationally tractable) because potentially greater compactness can be achieved and it can be done in a single step as opposed to trying numerous combinations of central units as seeds. Furthermore, the use of widely available generic optimization software, such as GAMS, makes the solution method and computational steps more transparent and easily transferable to similar problem situations.

2.3-C. Modification of Existing Plan Methods

Nagel (1965) proposed a heuristic plan to redistrict a state with the most recent district plan as the starting point. The plan would make incremental changes. Either one base unit or two base units at a time would be transferred or swapped for one or two units in a different district. In the first run, two criteria needed to be met before any swap could occur. The resulting districts needed to be contiguous and the given objective function needed to be improved. The objective function incorporated three criteria. The first was the average population deviation from the ideal. The second was the total number of districts that contained a population beyond a predetermined limit. The final criterion in the objective function was the deviation in compactness from the starting districting plan. The only way for a swap to occur was for at least one of these criteria to be improved sans any negative effects on the other criteria (Nagel, 1965). When no additional swaps or transfers were possible, the model proceeded to stage two, which considered political criteria. The same types of swaps were made in an attempt to improve on whatever the given criterion was. Swaps in the second stage could negatively affect the criteria from stage one. As in all other heuristics, the number of possible switches and swaps between all districts can be enormous and may be prohibitive when using this approach. For a given districting plan, it is possible to formulate the switch and swap method using a formal optimization approach where specified districting criteria are met and an objective function is optimized. The latter may be defined depending on the purpose. For instance, the total number of switches and swaps made can be minimized (thus minimizing the changes made to the existing plan) or the compactness of the resulting district configurations can be maximized. This is not done in the present thesis, but the model developed here can be modified easily to determine optimal district configurations by performing switches and swaps.

2.3-D. Geographical Methods

Forrest (1964) introduced a geographic method for dividing a state into equally populated districts. He called his method the method of diminishing halves and claimed the process “proceeds to slice up the state into…compact, contiguous, and equal population districts – plus or minus one half of 1%.” The method attempts to divide the state in half to create two approximately equally sized and populated areas. Each half is split according to the same standard until districts are formed. No explicit algorithm is given to determine where the lines should be drawn to divide each half. In addition, this method would be difficult to use if the total number districts is not even or divisible by a small integer (such as two or three). In the case of Illinois, for instance, the total number of congressional districts is 19, which is a prime number; therefore, the approach suggested by Forrest cannot be applied.

Spann et al. (2007) attempts to remedy the difficulty in Forrest's approach by defining a consistent process to determine where the dividing lines should be drawn. *Spann* uses the slope of the line perpendicular to the best-fit line of census tract centroids within the half to be divided with the hope of reducing the likelihood of dividing cities. The location of the line is chosen based on a position that divides the population in half. *Spann* concludes that the diminishing halves method performs worse in terms of compactness and community integrity than the moment of inertia method used by *Weaver and Hess* (Spann et al., 2007). However, the indivisibility problem, i.e. when the number of districts to configure is a prime number, remains.

Tobler (1973) incorporated an extremely creative change to an original districting map. A cartogram of the state was created. A cartogram is a graphical representation of a region creating base units that are proportional to a certain variable other than to land area (Tobler, 2004). The variable *Tobler* used was population. Therefore, any two units taking up an equivalent area in the cartogram contain the same population. Two geographic criteria were imposed to make districting possible once the transformed map was created. The total area of the state should be unchanged and any base units that were contiguous prior to the transformation should continue to be contiguous after the transformation. *Tobler* then was able to draw district boundaries based on creating districts of equal area in the transformed map. This is done by placing a grid over the transformed map with each square representing a district.

2.3-E. Recent Approaches to Redistricting

In the past ten to fifteen years a diverse new set of methods have been employed in the political districting problem to take advantage of advances in computing speed and to incorporate new ideas.

Mehrotra et al. (1998) use branch and bound techniques to generate districts in South Carolina. Districts are clustered around a center in a similar way to the *Weaver and Hess* method. The major inclusion in the article is that contiguity is guaranteed in the resulting district plan. They enforce this by requiring for any unit to be attached to a district center, at least one adjacent unit that is closer to the district center is also attached to that center. All units along the path to the center unit must also be attached to the center unit (Mehrotra et al., 1998). *Mehrotra's* model is similar to the model used in this thesis. The divergence is in *Mehrotra's* consideration of compactness. I use the population-weighted distance of each district as a proxy for compactness. Mehrotra also uses a population-weighted distance measure of compactness as well. However, the distance between units is calculated as the number of indivisible base units one must pass through to get from one unit to another unit. This relative measure of compactness attempts to correct for biases between rural and urban areas caused when using an absolute measure of compactness.

Galvão et al. (2006) use a method similar to *Mehrotra* to create contiguous districts called voronoi diagrams. The nearest neighbors are attached to create districts. The boundaries can be adjusted to allow for given criteria to be met.

Other recent methods of political redistricting have taken commonly used criteria and attempted to create district plans using approaches from statistical physics (Chou & Li, 2006), and by applying genetic algorithms (Bação, Lobo, & Painho, 2005).

From the description of the numerous methods discussed above, it should become clear that there is no clear best method for districting. Certain methods are more applicable than others to different types of problems.

2.4 P-Center Formulation

This section is a review of the general form of the mathematical technique used to create the political districting model put forth in this thesis. The p-center formulation, clustering, and connectivity are discussed.

The *p*-center formulation was used to answer the facility location (or salesman) problem. The facility location problem poses the question: How to select *p* facilities from a set of *m* possible locations for facilities and assign *n* clients in a way to minimize the maximum distance from a client to the facility he or she is assigned? The *p*-center formulation has been adapted to solve many problems including the school districting problem, where students must be assigned to certain schools with the goal of minimizing the distance children must travel to school while following given constraints; possibly taking into consideration the gender or racial composition of each school. The formulation has also been adapted to reserve design problems. The *p*-center formulation can be generalized to selecting *p* centers from a set of *m* possible centers and attaching units with definable characteristics to these centers in a way that minimizes the distance from units to centers also taking into consideration the characteristics of the units that will be assigned to each center.

2.4-A. Clustering

Clustering refers to assigning the assigning units that are closely grouped around the center to which they are assigned. This is also known as compactness[6]. Minimizing the total

[6] See *Williams et al.* (2005) for alternative ways to measure compactness and a summary of modeling methods that improve compactness of conservation reserves.

distance units are away from the assigned center tends to cluster units. Adding characteristic requirements to the p-center formulation tends to reduce clustering around centers. For example, in the school districting problem, requiring each school to be comprised of at least a given percent of a minority group would add a characteristic requirement to the formulation. This added requirement means that some students need to travel further away to their assigned school i.e., the students attending each school would be less clustered around the school with this requirement

2.4-B. Connectivity

Connectivity or adjacency of two spatial units occur when they share at least one common boundary. Modelers use graph theory to guarantee connectivity of units. The same idea of adding characteristic requirements to the p-center formulation can be utilized here. The characteristic of the unit is whether it shares a boundary with another unit or not. Given set of units that are all assigned to a given center, the set is considered fully connected if every unit shares at least one boundary with another unit and at least one unit shares a boundary with the center (for mathematical programming formulations of connectivity, see Önal and Briers, 2006; Cerdeira et al., 2005). Clustering tends to assign units that are also connected, but this result is not guaranteed. A set of units that is not fully connected is fragmented. Fragmentation is not allowed in political districting.

Chapter 3: A Mathematical Programming Approach to Political Redistricting

A general mathematical programming approach to political districting is presented. The approach is an augmentation of the p-center formulation that uses clustering to create compact districts.

3.1 Basic Units and District Centers

The standard procedure for a districting problem is to assume the area of interest, whether it is a state that needs to be apportioned into congressional districts or a city that needs to be divided into wards, is wholly comprised of i indivisible units. Units can be census tracts, counties, or any other geographic entity. Each unit must be assigned to one and only one of the districts. The area of interest contains a total of k districts and because units are indivisible, the number of units must meet or exceed the number of districts in the particular area of interest $i \geq k$. A district plan is the set of k districts that encompass the area of interest.

As mentioned in earlier chapters, political districts must be contiguous. The model presented here does not explicitly guarantee contiguity, but it uses compactness as a tool for promoting contiguity. More specifically, the model selects a given number of central units and attaches a set of closest units to each central unit to create districts. It is rare that the contiguity requirement is violated. If it is the case, corrections can be made ex post facto.

Every unit has a population (p_i) and a location expressed by the latitude and longitude of centroid of the unit. Using the great circle distance formula, it is possible to create a parameter consisting of the distances (d_{ij}) from every unit i to every other unit j , where j is a duplication of the set of i units.

Let c denote the set of possible district centers. A sufficiently large[7] subset (c) of units is chosen from i units ($c \subset i$). The set c contains numerous units spread over the area of interest. The model chooses k units from c to serve as district centers. Allowing the model to endogenously choose the optimal centers is in contrast to *Weaver and Hess*' a priori selection of the exact centers.

3.2 Optimization Procedure

The objective function of the optimal redistricting model is to minimize the total population weighted distance to each district center summed across all the districts. This can be alternatively described as the total distance traveled for every person in the area of interest to reach the centroid of his/her respective district. The population-weighted distance is used here instead of population-weighted moment of inertia as in *Weaver and Hess* (1963). Define X_{ic} as a binary value taking a value of 1 when unit i is attached to center unit c and 0 otherwise. The objective function of the districting model is then given by:

$$\text{Minimize} \sum_{i,c} p_i d_{ic} X_{ic} \tag{3.1}$$

3.2-A. Model Constraints

Constraints must be added to guarantee that the model behaves as intended and generates a desirable district configuration. First, a constraint is included to insure that the proper number of districts is formed. Equation (3.2) forces exactly k district centers to be chosen from the possible district centers.

$$\sum_{c} X_{cc} = k \tag{3.2}$$

[7] larger than the total number of districts, and preferably as large as possible

As stated previously, every unit must be part of exactly one district. Equation (3.3) reflects this requirement in the model.

$$\sum_c X_{ic} = 1 \text{ for all } i \tag{3.3}$$

The unit assignment to district centers is governed by the following constraint:

$$\sum_i X_{ic} \le gX_{cc} \text{ for all } c \tag{3.4}$$

where g is a sufficiently large scalar whose value is the maximum allowable number of units that may comprise a district. Constraint (3.4) requires that a unit can only be assigned to a possible district center if the latter is chosen as one of the k district centers. As an example, suppose the first possible center, $c = 1$, is not chosen as one of the k selected actual district centers. This means $X_{11} = 0$. Plugging $X_{11} = 0$ into the constraint results in the right-hand side equaling zero. The only way for the inequality to hold true is for the left-hand side to also equal zero $\left(\sum_i X_{i1} = 0\right)$. Remember that X_{i1} is a binary variable taking the value of 1 when unit i is attached to the first center. So, $\sum_i X_{i1} = 0$ means that no unit is attached to possible center 1 when $c = 1$ is not a selected center. Alternatively, when $c = 1$ is chosen as one of the k district centers the model allows attaching up to g units to c to form a district around that center.

Equations (3.1) - (3.4) form the general compact districting model.

$$\text{Minimize} \sum_{i,c} p_i d_{ic} X_{ic} \tag{3.1}$$

Subject to:

$$\sum_c X_{cc} = k \tag{3.2}$$

$$\sum_c X_{ic} = 1 \text{ for all } i \tag{3.3}$$

$$\sum_i X_{ic} \le g X_{cc} \text{ for all } c \tag{3.4}$$

The general districting model creates compact districts. It does not however take into consideration any other common redistricting criteria.

3.2-B. Inclusion of Optional Criteria

This section will add to the general compact districting model by imposing some common constraints discussed in Chapter 2. Included constraints insure population equity, community integrity, voting results proportional election results, and minority districting. Any number of these new criteria can be added to the general compact districting model resulting in a new districting outcome that satisfies the needed requirements[8].

Population equity is a requirement in the United States implemented by *Reynolds v. Sims* (1964) and *Wesberry v. Sanders* (1964). The issue with population equity that has often been left for interpretation is how much of a deviation from the ideal (average) district population is acceptable. In the providences of Canada, districts may deviate by as much as 25% from the ideal district population though most deviate by less than 15% (Courtney, 2004). In Kentucky and many other states, the maximum allowable deviation from ideal is 5% (Fischer v. State Board of Elections, 1994). In other states, there is not a clearly defined deviation limit. Regardless of the chosen allowable deviation limit, constraints can be imposed to require population equity.

$$\sum_i p_i X_{ic} \le (1+l) * \left[\frac{\sum_i p_i}{k} \right] * X_{cc} \quad \text{for all } c \tag{3.5}$$

[8] Or will show that a district plan is not feasible with the given criteria.

$$\sum_i p_i X_{ic} \geq (1-l)*\left[\frac{\sum_i p_i}{k}\right]*X_{cc} \quad \text{for all } c \qquad (3.6)$$

Let l be the maximum allowable deviation above or below the ideal district population. The left-hand side of constraint (3.5) and (3.6) is the total population for district c. The right hand side of the first inequality is the maximum district population allowed. Reading the first constraint in its entirety, the actual district population must be less than or equal to the maximum allowed district population. Constraint (3.6) uses the same logic. Constraints (3.5) and (3.6) can be added to the general compact districting model to require population equity based on the extreme deviation measure.

Some states legally discourage the division of communities. This is often manifested in the requirement to divide as few counties as possible, but it could be a requirement to divide cities or towns as seldom as possible. The community integrity criterion requires some alterations to the general compact districting model.

Let Y_{cq} be a binary variable equal to 1 when any unit within community q is attached to center unit c and 0 if no unit within community q is attached to center unit c. An example will make this more transparent. Assume the first community $(q=1)$ is comprised of two units. If both units within that community are attached to district center 1, then $Y_{11}=1$ and $Y_{21}=0$. $\sum_c Y_{cq}$ for that particular community, would be 1 meaning that community one is completely contained within one district. Now assume the same scenario except that one unit in community 1 is attached to district center 1, while the other unit in the community is attached to district center 2. In this case, $Y_{11}=1$ and $Y_{21}=1$. $\sum_c Y_{cq}$ for this community would be two meaning community 1 is divided between two districts.

In order to incorporate community integrity, the model must minimize the total number of districts that all communities belong to (Y_{cq}). This can be incorporated into the general compact districting model by adding a second term to the objective function, which makes the model a multi-objective programming model, and adding two additional constraints. The objective function will now be:

$$\text{Minimize} \sum_{i,c} p_i d_{ic} X_{ic} + s \sum_{c,q} Y_{cq} \tag{3.7}$$

where s is a weight (penalty) factor that reflects how important dividing communities is relative to compactness. The value of s is chosen a priori and given an increasing value as community integrity increases in importance relative to compactness.

Two constraints are also needed to ensure that the binary variable Y_{cq} behaves as intended. These are:

$$\sum_{i} X_{ic} \leq gY_{cq} \text{ for all } c \text{ and } q \text{ where } i \text{ is within community } q \tag{3.8}$$

$$\sum_{q} Y_{cq} \leq gX_{cc} \text{ for all } c \tag{3.9}$$

Constraint (3.8) stipulates that $Y_{cq} = 1$ only when some unit in community q is attached to a particular district center ($X_{ic} = 1$).

Constraint (3.9) implies that $Y_{cq} = 1$ only when a possible district center is actually chosen as a district center $(X_{cc} = 1)$. If a candidate center is not chosen $(X_{cc} = 0)$, the right-hand side of that constraint is 0. The only way for the inequality to hold true when the right-hand side equals zero is for the left-hand side to also equal zero $(Y_{cq} = 0)$. Altering the objective function and including the two new constraints together impose the criterion of community integrity.

Another goal that a legislature or committee might want to implement in a redistricting plan is the criterion that election results closely reflect voting results. Consider a country with two political parties and 100 representative seats. If 60% of the general population votes for party A and the other 40% vote for party B, it would seem most equitable to have 60 party-A representatives and 40 party-B representatives. Since voting is done at the district level in the United States, this equitable outcome is often distorted and while it may not be possible to reach absolute equity a tolerance level may be imposed.

Additional constraints need to be incorporated to the general compact districting model above to require that election results are similar to voting results. Let v_{iw} be denote the number of people in unit i who voted for party w. It would be preferable to get the average voting results over an extended period to time to predict future voting results. Let E_{cw} be a binary variable taking the value of 1 when political party w wins the district with center unit c and 0 otherwise.

$$\sum_{w} E_{cw} = X_{cc} \text{ for all } c \tag{3.10}$$

Equation (3.10) requires that exactly one representative wins every district. If the candidate center unit c is not chosen as an actual center of a district ($X_{cc} = 0$), no representative can be elected there. If the candidate unit c is chosen to be a district center $(X_{11} = 1)$, then either $E_{1A} = 1$ or $E_{1B} = 1$. In the first case party A wins the election in this district ($E_{1A} = 1$), and party B by default did not win, thus $E_{1B} = 0$. The opposite occurs if $E_{1B} = 1$.

The following constraint determines the winning party for a given district:

$$\sum_{i} v_{iw} X_{ic} - .5\left(\sum_{i,w} v_{iw} X_{ic} \right) \le hE_{cw} \text{ for all } c \text{ and } w \tag{3.11}$$

where h is an arbitrary large number[9]. Constraint (3.11) defines the link between voting results and election results. This constraint articulates that when a particular party gets more than 50% of the votes in a district, that party wins the election ($E_{cw} = 1$).

Constraints (3.12) and (3.13) incorporate the allowable tolerance for election results to vary from voting results.

$$\frac{\sum_{c} E_{cw}}{k} \leq (1+t) * \left[\frac{\sum_{i} V_{iw}}{\sum_{i,w} V_{iw}} \right] \text{ for all } w \tag{3.12}$$

$$\frac{\sum_{c} E_{cw}}{k} \geq (1-t) * \left[\frac{\sum_{i} V_{iw}}{\sum_{i,w} V_{iw}} \right] \text{ for all } w \tag{3.13}$$

The left-hand side of these two constraints is the total number of districts won divided by the total number of districts. This is the percent of districts won by party *w*. The bracketed piece of the right-hand side of each constraint is the total number of people that voted for party *w* divided by the total number of people that voted, thus the percent of the population that voted for party *w*. The constraint means that the percent of districts won by a party must be within the tolerance (t) above or below the percent of total votes received by that party. Constraints (3.10) - (3.13) can be added as constraints to the general compact districting model to force election results to be similar to aggregate voting results.

Proportional representation of minorities is another consideration that may be undertaken to provide fair representation to minorities. The best way to explain this concept is to give an example. Consider a state consisting of only two groups of people, one group is majority and the other is minority. Assume that the minority population of this state is 30%, thus the majority

[9] using the total population size would work well here.

group comprises 70% of the total population. If the population is distributed within each district in equal proportion to the state percentages and voters in the majority group vote for a majority candidate, then minority candidates would lose in every district because they will never have greater than 50% of the votes needed to win an election. A way to correct this problem is to make 30% of the districts within the state majority-minority districts. A majority-minority district has a population composed of at least 50% minority, and result in the opportunity for the minority group to have representation that is proportional to the composition of the state.

Let m_{iz} be a parameter representing the number of each type of person that makes up each unit i and z be the set of groups of people[10]. The constraints to insure proportional representation of minorities is a slight variation of the constraints needed for the election results to equal the voting results. Instead of having two political parties, there are two groups of people now, the minority and the majority groups. Let R_{cz} be a binary variable that takes a value of 1 when a candidate from group z wins an election in district *c*. The following constraints insure that among the districts created by the model, either a minority representative or a majority representative wins, but not both:

$$\sum_z R_{cz} = X_{cc} \text{ for all } c \tag{3.14}$$

$$\sum_i m_{iz} X_{ic} - .5\left(\sum_{i,z} m_{iz} X_{ic}\right) \le hR_{cz} \text{ for all } c \text{ and } z \tag{3.15}$$

Equation (3.14) implies that if c is not selected as a district center, i.e. $X_{cc} = 1$, then $R_{cz} = 0$ for both groups; and conversely if $X_{cc} = 1$, then one and only one of $R_{cz} = 1$. Constraint (3.15) is very similar to constraint (3.11). It creates the linkage between

[10] z would have been pirates and maidens in the previous example.

minority/majority makeup of each district and the election result. Whichever group makes up greater than 50% of the population in a district wins that district.

$$\frac{\sum_{c} R_{cz}}{k} \leq (1+t)*\left[\frac{\sum_{i} m_{iz}}{\sum_{i,z} m_{iz}}\right] \text{ for all } z \tag{3.16}$$

$$\frac{\sum_{c} R_{cz}}{k} \geq (1-t)*\left[\frac{\sum_{i} m_{iz}}{\sum_{i,z} m_{iz}}\right] \text{ for all } z \tag{3.17}$$

Constraints (3.16) and (3.17) are counterparts of constraints (3.12) and (3.13). These constraints impose the criteria that the percentage of districts won by a minority group must be within a predetermined deviation of the percent of minorities in the population.

This section introduced the general compact districting model as well as multiple criteria that may be added including population equity, community integrity, and having the election results reflect the political or minority/majority makeup of the population to serve the interests of legislatures or committee members that are using the model.

An important issue to note is that using computers and optimization techniques does not result in just one map that eliminates human interaction. The model can produce many districting maps adapted to follow any criteria that are relevant for a given districting problem. The model has its real power in the ability to determine what changes would occur if a new criterion was added or even if a districting solution is possible that fits all of the proposed criteria. Computer aided districting methods can be an added tool in the redistricting process to help people quickly gain information and allow them to make comparisons between different plans.

Chapter 4: Application of the Model to Kentucky State Senate Districting

This chapter presents empirical results of the model described in Chapter 3 to the Kentucky State Senate districting problem. The Senate districting problem is chosen as a case study here to illustrate the workings of the model and investigate the effects of alternative districting maps on some socio-economic indicators[11]. With proper modifications (number of districts, average population by districts, etc.) the same approach can be used for any other political districting problem including House of Representatives, Senate, and Congressional districts.

Kentucky has 120 counties including 994 census tracts that must be grouped in 38 state senate districts. When configuring the senate districts, besides the standard contiguity and population equity requirements, the state constitution also requires minimum possible division of communities, namely counties. While most counties are sparsely populated and need not be divided, three densely populated counties have to be divided into several districts in order to achieve almost equal population distribution among districts. The model application presented in this chapter addresses all these considerations and determines a districting map that divides the Kentucky counties only when necessary and in a minimal way. The results obtained from the model are then compared with the Kentucky Senate districting map that has been in effect for the past few elections.

4.1 Kentucky Redistricting Problem

In 1993, a resident of Kentucky, Joseph Fisher filed suit against the Kentucky State Board of Elections. He claimed that the redistricting of the state House of Representatives and the state Senate districts violated Section 33 of Kentucky's Constitution. Section 33 of the Kentucky Constitution states:

[11] See Chapter 5

"The ... General Assembly after the adoption of this Constitution shall divide the State into thirty-eight Senatorial Districts, and one hundred Representative Districts, as nearly equal in population as may be without dividing any county, except where a county may include more than one district…the counties forming a district shall be contiguous"

Out of 120 counties in Kentucky, most counties were not divided. They were included in their respective districts as a whole. However, the proposed redistricting plan divided the three densely populated counties, Fayette, Jefferson and Kenton, between 19 Senate districts[12].The Supreme Court concluded that, "…as between competing concepts of population equality and county integrity, the latter is of at least equal importance." The Court officially interpreted that Section 33 requires redistricting plans to create districts within a 5% deviation above or below the average district population and divide the counties as minimally as possible. No one was sure of what the minimal amount of county divisions should be, but a new and improved redistricting plan was introduced in 1996, the plan was updated again in 2002[13]. The newest districting map reduced the divisions from 19 to 15, which was a significant improvement in terms of county divisions. Whether the current map is the best possible configuration is still unknown. This issue is addressed here and I will show that there is room for further improvement. Following the interpretation of the Supreme Court, the general compact districting model developed in Chapter 3 will be applied to the Kentucky data and a state Senate redistricting plan that has fewer community divisions will be generated while satisfying all of the criteria set out in the Kentucky Constitution.

[12] The total number of divisions of all counties between the House of Representatives districts was 48.

[13] Most likely a new plan will be introduced based on the 2010 census results.

The Kentucky State Senate has 38 seats; therefore, the districting model will configure 38 districts. First, a relatively large number of potential district centers are identified and the model is instructed to choose 38 district centers from the list of possible centers. The closest units are assigned to each district center in such a way that: i) the total population of every district shall be within 5% above or below the average district population, and ii) counties must be divided as minimally as possible. The next section will explain the process of implementing the general districting model to follow these mandates.

4.2 Kentucky Data

Census tracts are used as the 'indivisible' units, in this application[14]. Census tracts are geographic units created by the U.S. Census Bureau used to display statistical data. Census tracts are always nested within a county, meaning that a tract never crosses county borders. They are more or less similar in terms of population size, socio-economic characteristics, and living conditions of the inhabitants. Figure 3 displays the 994 census tracts for the state of Kentucky. Figures 4-6 display the census tracts of the three highly populated counties, Jefferson County, which contains Louisville, Fayette County and Kenton County respectively.

[14] This is a simplifying assumption. Although almost all tracts were assigned to their districts as a whole (without dividing), a few tracts were divided in the actual plan.

Figure 3: Kentucky Map of Census Tracts

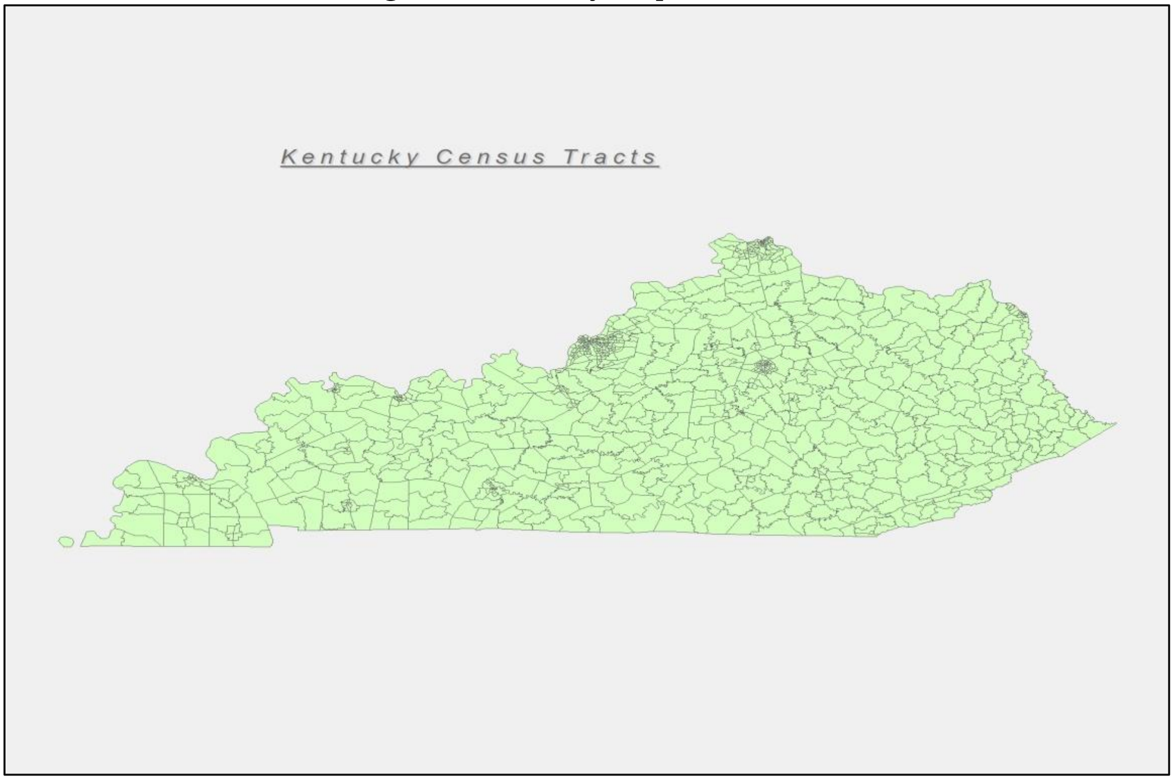


Figure 4: Jefferson County Census Tracts

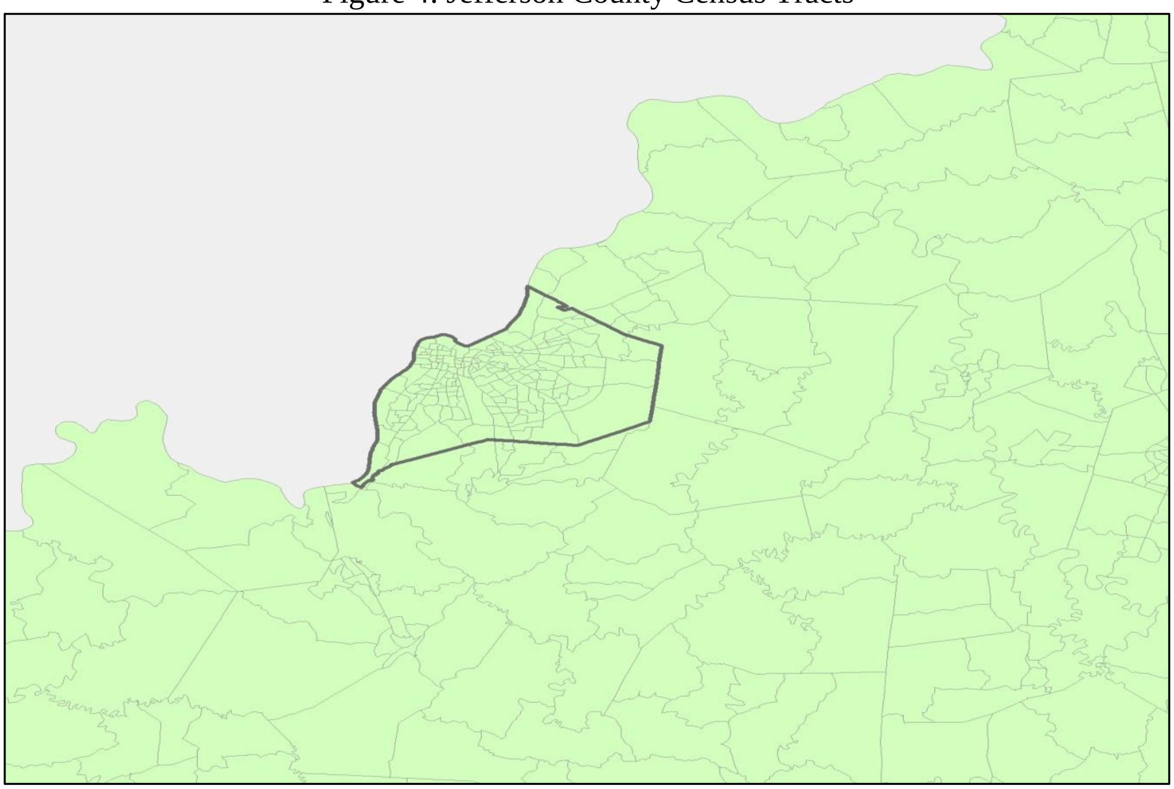


Figure 5: Fayette County Census Tracts

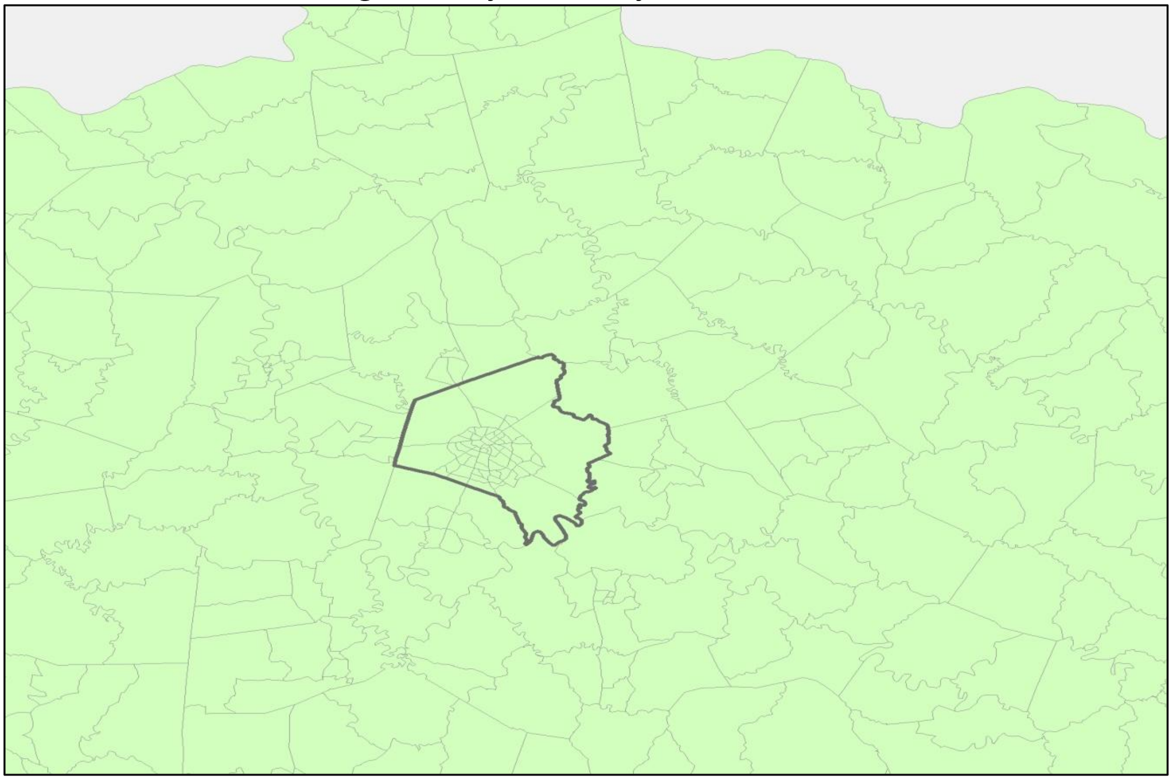
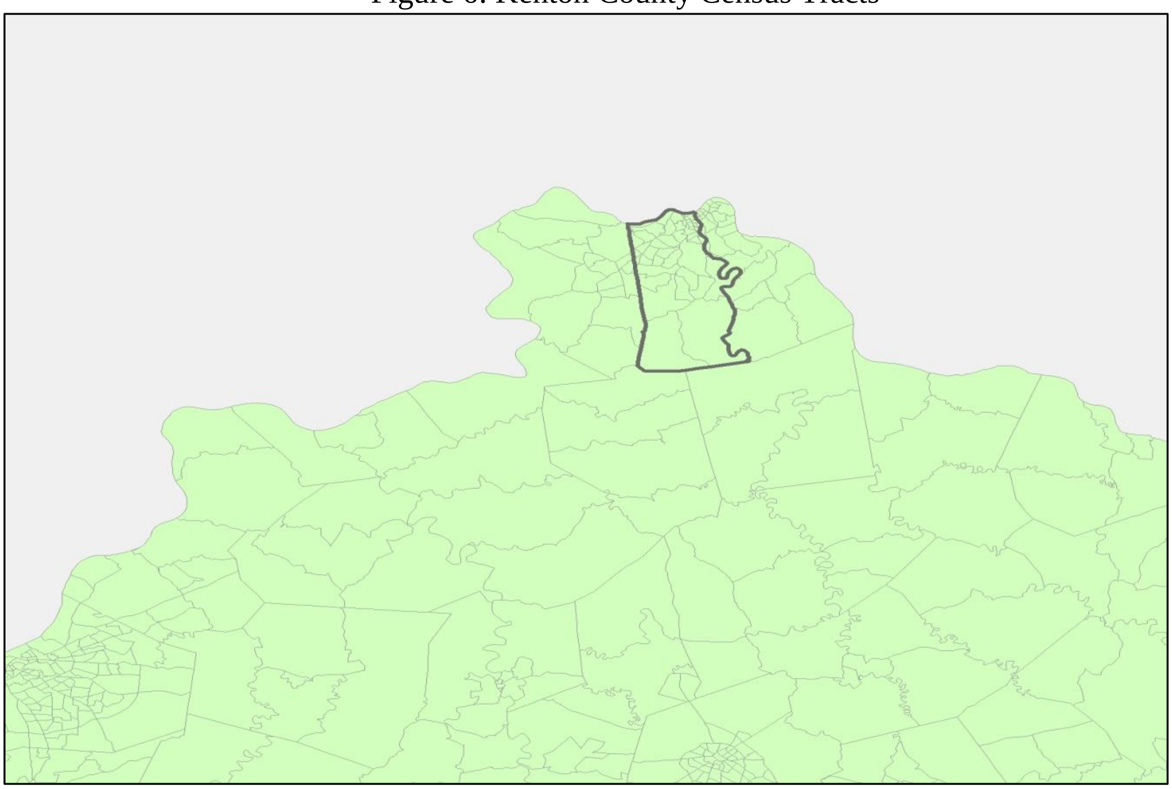


Figure 6: Kenton County Census Tracts



Using the 2000 U.S. Census gazetteer files, the data on the population, longitude, and latitude of all 994 census tracts in Kentucky were obtained. As of 2000, there were 4,041,769 residents of Kentucky. The population of each tract is assumed to be concentrated at one point in that tract, namely the centroid of the tract, whose latitude and longitude are given in the data. The average census tract in Kentucky had 4,066 people. The largest census tract had 15,386 people and the smallest census tract had 25 people. The Tract ID number uniquely identifies each census tract. The first two numbers indicate the state the census tract belongs to (Kentucky's is 21). The following three numbers identify the county (001 is Adair County). Kentucky has one-hundred and twenty counties. The final six numbers in the tract ID code is the census tract number (970200 is the census tract number in Figure 7). The six-digit census tract number is not unique and therefore cannot be separated from the rest of the tract ID code.

Figure 7: Example of U.S. Census Gazetteer Data

Tract ID	Population	Latitude	Longitude
21001970200	1497	37.178	-85.323

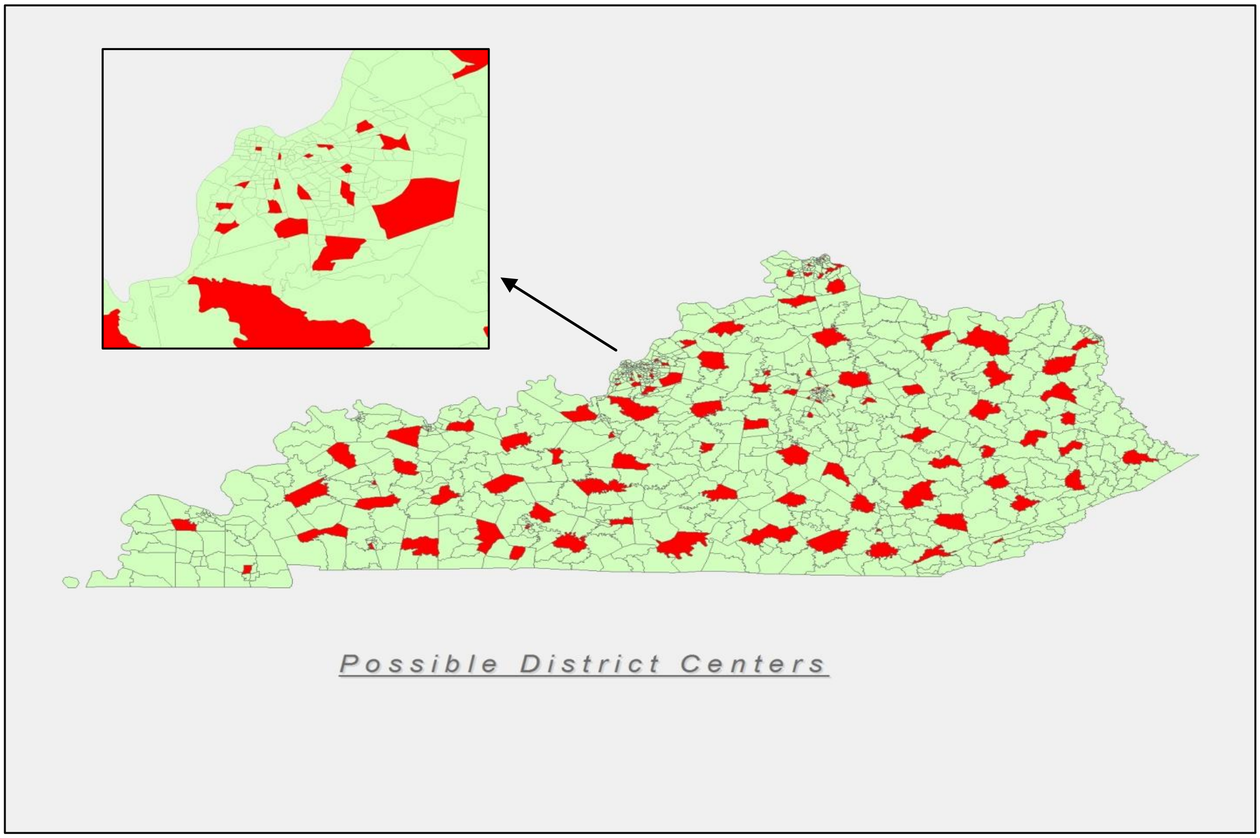
Using the latitude and longitude of every census tract and the great circle distance formula, the distances (d_{ij}) from every census tract (i) to every other census tract (j), where j is a duplication of the set of i census tracts, are computed. These distances are used when grouping tracts around a central tract as explained in Chapter 3.

I use maps created in ArcGIS 9 to display my results. I downloaded a map of Kentucky containing the census tract boundaries from the US Census Topologically Integrated Geographic Encoding and Referencing system (TIGER). I added as an attribute to each census tract the association between census tract and district center found as the solution to the optimization model. Finally, I aggregated and formatted census tracts by their district association to create the resulting maps found in this chapter.

4.3 Model Simplifications

It would be preferable to allow any unit (tract) be a possible district center as this would lead to consideration of all possible district configurations and select the best from among those. However, this would increase the number of binary variables in the model, which in turn would increase the computational complexity since integer programming models become harder to solve when the model size, particularly the number of binary variables, gets larger. In this particular application consideration of all 994 tracts as potential district centers would require enormous computing power (particularly memory) and processing time needed to solve the problem, therefore computational complexity would be a serious bottleneck. Instead of considering all tracts as potential district centers, a much smaller subset is used for this purpose. Specifically, from the 994 census tracts, only 100 were chosen to be possible district centers (which are denoted by c in the algebraic model described in Chapter 3). Given that only 38 of them will actually be selected as centers in the solution, considering 100 tracts is thought to provide sufficient flexibility for selection. Furthermore, in order to avoid spatial bias, these possible centers are geographically spaced almost uniformly throughout the state. None of the tracts near the state boundary are allowed to be a center. Finally, a higher concentration (much closer centers) is assumed around densely populated areas. Figure 8 displays the set of potential district centers considered in the model.

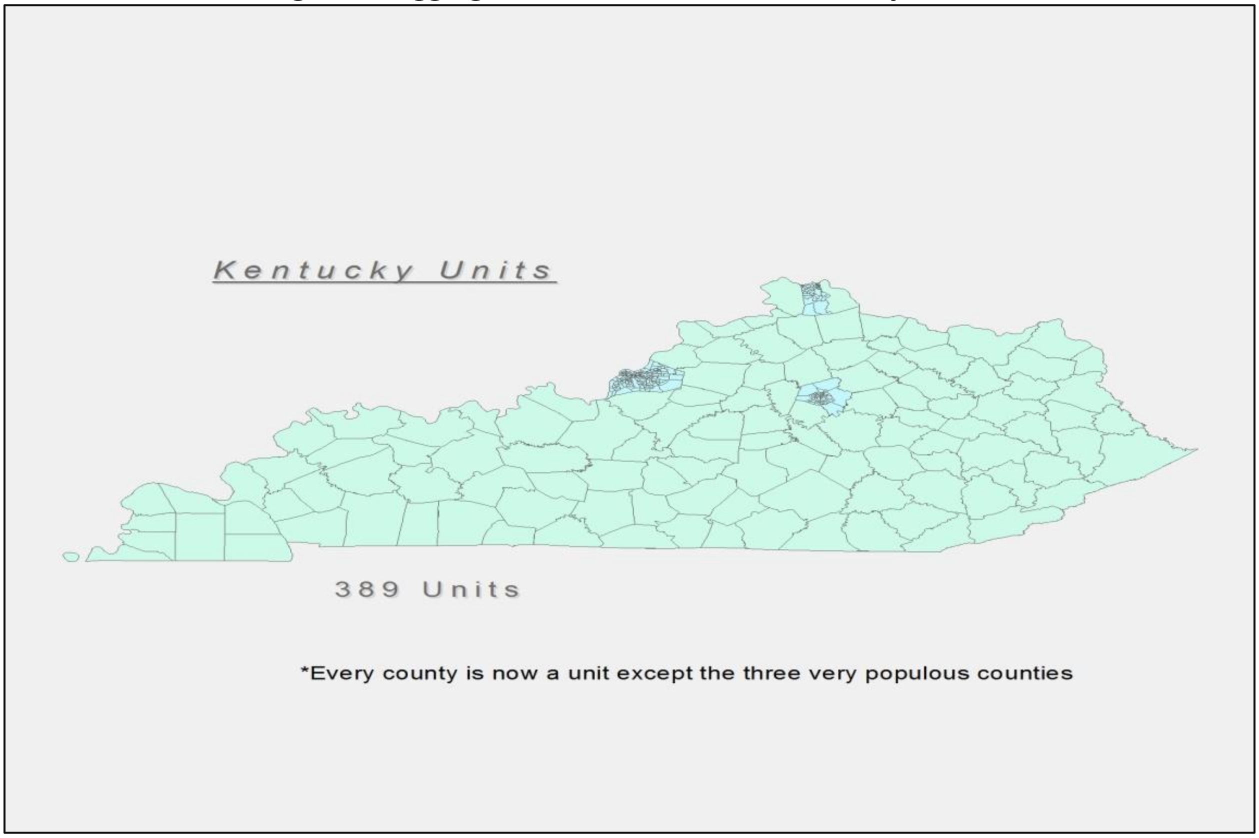
Figure 8: Possible District Centers Allowed in the Model



With a state population of 4,041,769 and 38 districts to be created for the state senate, the ideal district population is calculated as 106,362. There are three counties in Kentucky each containing a great enough population to make up more than one district. According to the Kentucky Constitution, this is the only allowable reason for a county to be divided. Jefferson County has 693,604 residents and contains the city of Louisville. Fayette County has 260,512 residents and is home to the city of Lexington. The final county with a substantially large population is Kenton County, which has 151,464 inhabitants and is just south of Cincinnati.

The set of indivisible units is redefined in order to allow the model to divide only these three densely populated counties. Any county other than Jefferson, Fayette, and Kenton is assumed as an indivisible unit, as they are treated in the actual districting process. The population of each of these units is assumed to be concentrated at the location of the centroid of the census tract with the greatest population within that county. For the three highly populous counties, the census tracts that comprise them are considered indivisible units (i). This aggregation of census tracts to the county level reduces the total number of units from 994 to 389 (See Figure 9).

Figure 9: Aggregation of Census Tracts to County Level



4.4 Description and Explanation of the Mathematical Model

The mathematical model introduced in Chapter 3 considers compactness as the only spatial criterion when aggregating base units into districts. As elaborated earlier, this criterion is actually intended to promote configuration of contiguous districts, but in the process, the districts become compact as well. This criterion is used in the objective function of the integer programming model where the sum of distances between selected district centers and the units assigned to them is minimized, which automatically establishes (in most cases) spatial contiguity. The population equity criterion is modeled in a separate constraint. As mentioned in the above discussion, besides contiguity and population equity, minimal division of counties has to be incorporated as a third explicit requirement in Kentucky districting. Since the minimal number of divisions is unknown, the only way to address this issue is to include divisions (to be minimized) in the objective function. Therefore, the model must consider two objectives, namely compactness of districts and the number of county divisions. In general, these two objectives are not compatible, that is it may not be possible to minimize county divisions and maximize compactness at the same time. This is because a more compact district configuration can be obtained by dividing counties, since dividing large shapes (counties) and aggregating them in an appropriate way would lead to more circular district shapes. Thus, the problem at hand is a typical multi-objective optimization including two objectives that conflict with each other. Moreover, these objectives are expressed in different units (specifically miles and number of divisions); therefore putting them together in a unified objective function requires a conversion. A widely used multi-criteria optimization method for such cases is to introduce different weights for individual objectives and create an objective function, which is a linear combination, the individual objectives (sum of weights times the objective functions). Since two objectives are involved in this particular case, only one weight suffices. This leads to an objective function defined as the sum of distances from all tracts/counties to their assigned district centers (both district centers and unit assignments are to be determined simultaneously) and a constant multiple of the total number of divisions. The multiplier assigned to total county divisions can be interpreted as the importance of community divisions relative to compactness of the districts. Therefore, if a large weight is assigned to county divisions less compact districts may be configured while the total number of divisions is little. If, on the other hand, a small weight is assigned to county divisions more divisions may occur in the solution while compactness of districts is improved.

The explicit mathematical model applied to the Kentucky State Senate redistricting problem is described below.

4.4-A. Explicit Mathematical Model for Kentucky Senate Districting

$$\text{Minimize} \sum_{i,c} p_i d_{ic} X_{ic} + s \sum_{c,q} Y_{cq} \tag{4.1}$$

Subject to:

$$\sum_{c} X_{cc} = 38 \tag{4.2}$$

$$\sum_{c} X_{ic} = 1 \text{ for all } i \tag{4.3}$$

$$\sum_{i} X_{ic} \leq 50 X_{cc} \text{ for all } c \tag{4.4}$$

$$\sum_{i} p_i X_{ic} \leq (1+.05) * \left[\frac{\sum_i p_i}{k} \right] * X_{cc} \quad \text{for all } c \tag{4.5}$$

$$\sum_{i} p_i X_{ic} \geq (1-.05) * \left[\frac{\sum_i p_i}{k} \right] * X_{cc} \quad \text{for all } c \tag{4.6}$$

$$\sum_{i} X_{ic} \leq g Y_{cq} \text{ for all } c \text{ and } q \text{ where } i \text{ is within community } q \tag{4.7}$$

$$\sum_{q} Y_{cq} \leq g X_{cc} \text{ for all } c \tag{4.8}$$

4.4-B. Specification and Explanation of the Mathematical Model

Minimization of the objective function implies that the model must choose an allocation of units to district centers in such a way that the closest units are attached as much as possible and the total number of county divisions is minimized. The relative importance between compactness and community division is determined by the value given to s. The larger the value of s, the greater the penalty for dividing a county. In this application a value of 5000 was chosen for s. This number reflects the relative importance given to compactness and county integrity.

Equation (4.2) requires that out of the 100 units that can possibly be district centers, exactly 38 will be selected in the solution. This is in line with the wording of Kentucky State Constitution requiring that the General Assembly should create 38 state Senate districts.

Equation (4.3) forces each unit (i) to be attached to exactly one district center. This implies that no tract can be part of more than one district, in other words districts have to be disjoint.

Constraint (4.4) mandates two important rules. First, if a possible center is not chosen as one of the 38 actual district centers, no units may be attached to it. The next piece of this constraint intends to simplify the complexity of the model and reduce computing time. When a possible center unit is chosen as one of the 38 district centers, at most 50 units may be attached to it. There are 389 units in the model and 38 districts are chosen, implying the average district center will have 10 units attached it. The large upper bound (50) is unlikely to affect the solution, but it reduces the number of binary variables representing unit assignments to districts by limiting the number of units that may be attached to a district center, thus it reduces the computing time.

Constraints (4.5) and (4.6) specify a range that the population of each district must fall within. The Supreme Court specified that the population of a district should be within 5% of the ideal district population. The ideal Kentucky state Senate district population is 106,362. Using the 5% deviation limit, the largest allowable district population is 111,680 and the minimum permissible district population is 101,044.

Constraints (4.7) and (4.8) govern the proper function of the Y_{cq} variable. The first imposes the condition that if any unit within a county is attached to a particular center c, i.e. $X_{ic} = 1$ for some i in county q, then $Y_{cq} = 1$ which means that county q has a part in the district centered at c. If $Y_{cq} = 1$ for some other c, then county q is divided and the sum of all such divisions is expressed in the objective function. Constraint (4.8) stipulates that if a possible district center is not chosen as one of the 38 district centers $(X_{cc} = 0)$, then $(Y_{cq} = 0)$ for all q, that is no county can have any piece within a district centered at c (because such a district is not configured). Conversely, if $Y_{cq} = 1$ for some q, then there must be a district centered at c and county q has a piece in that district, as it should be. These two constraints force the binary Y_{cq} variable to behave as intended. It takes the value of one when at least one unit within that county is part of a district centered at c.

One final criterion was added to ease model complexity and allow the model to be solved in a reasonable amount of processing time. In order to keep the model computationally tractable, for each possible district an area around that center containing twice the ideal (i.e. average) population was created. The model is allowed to choose units to be assigned to that center only within that area. This significantly reduced the model size, which in turn reduced the time needed to solve districting problem. The constraint had little effect on the actual resulting districting plan because out of the area of feasible units making up 200% of the ideal district population, an actual district must be chosen with between 95% and 105% of the ideal district population. Two examples of configured districts are shown in and Figure 11 and Figure 13.

Figure 10 shows the allowable units set in a sparsely populated area in southern Kentucky. Figure 11 shows the district created by the model in blue. The yellow base units are the set of units that make up 200% of the average district population around the red possible district center. Figure 12 shows the allowable set around a possible center in the densely populated Louisville area.

Figure 10: Twice the Average District Population Around a Potential Center in S. Kentucky

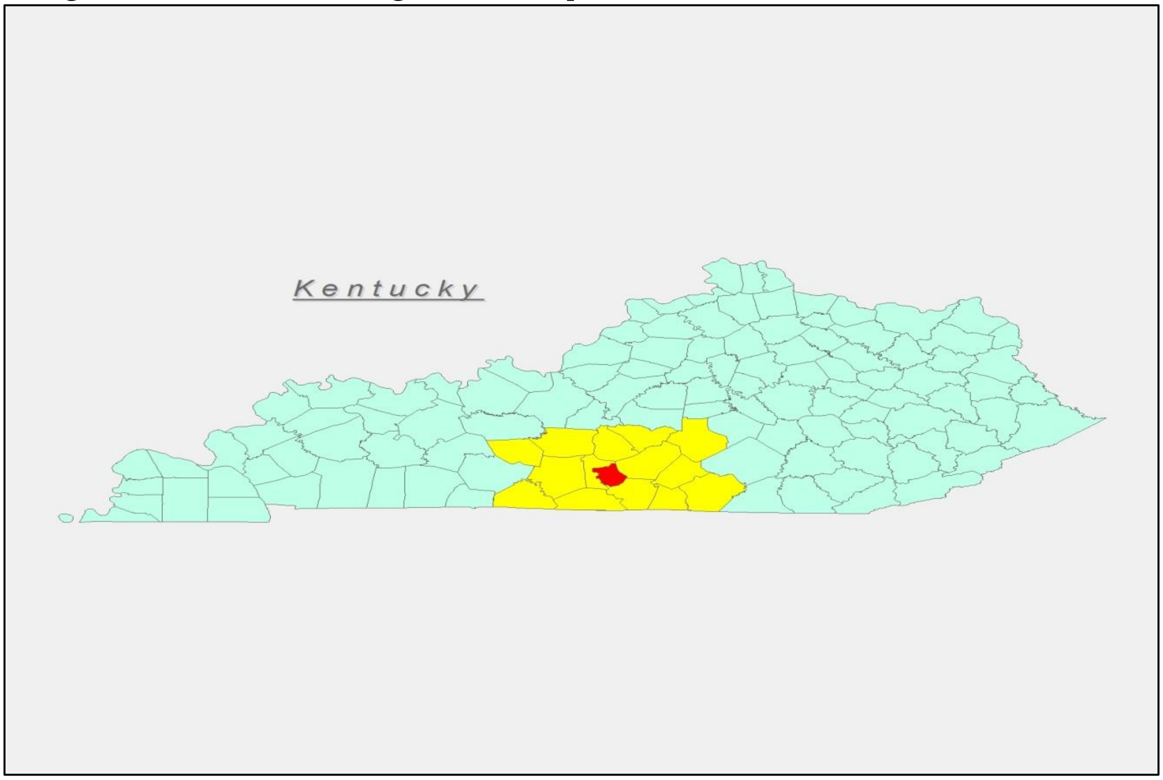


Figure 11: District Chosen Around Center

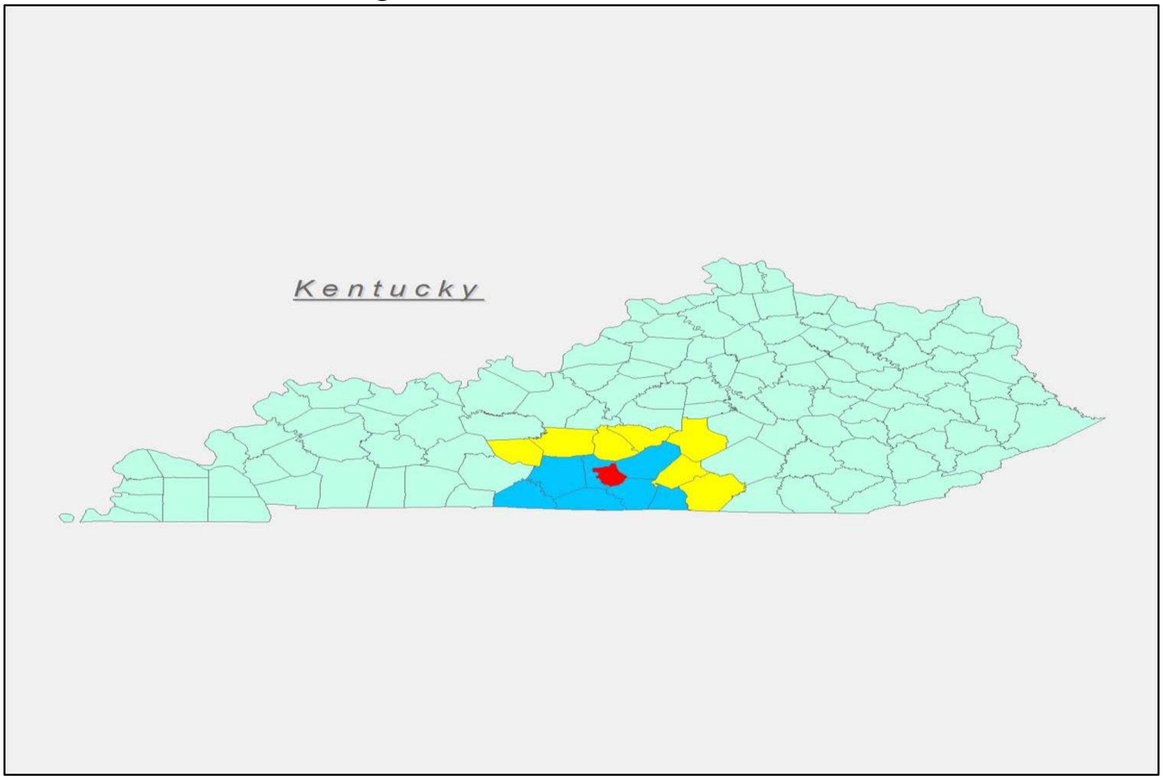


Figure 12: Twice the Average District Population Surrounding Center in Louisville

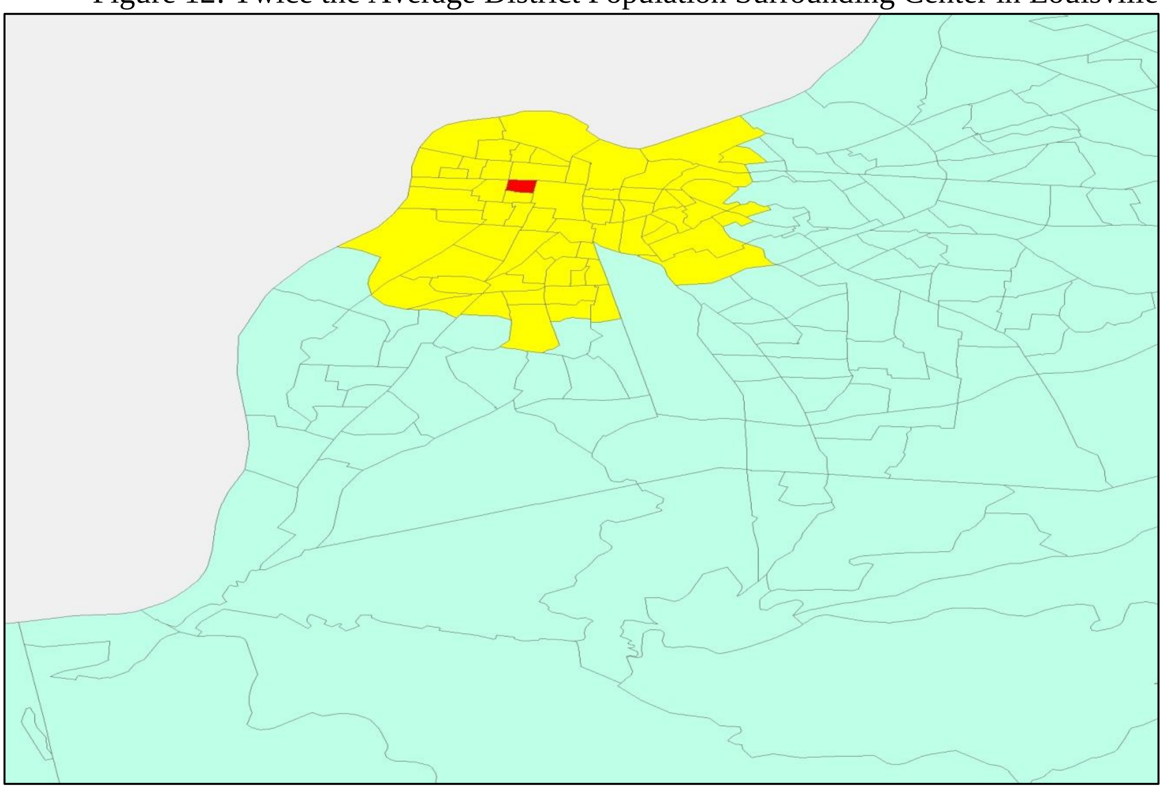
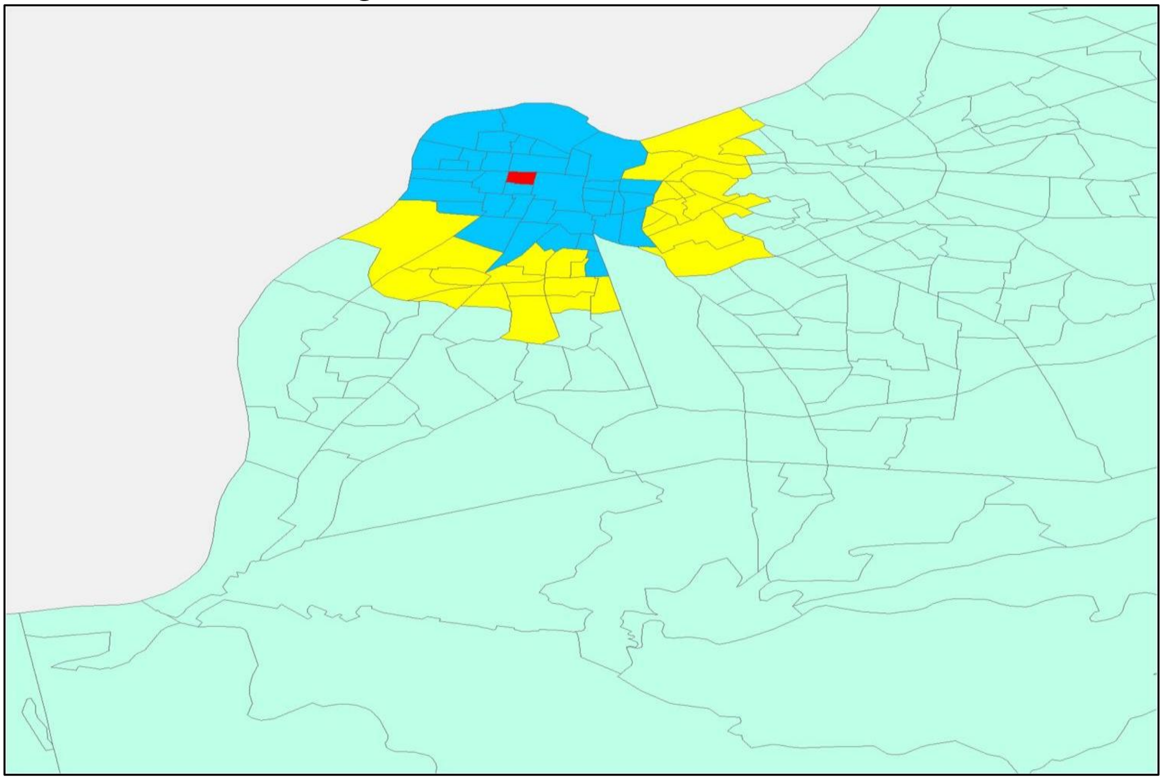


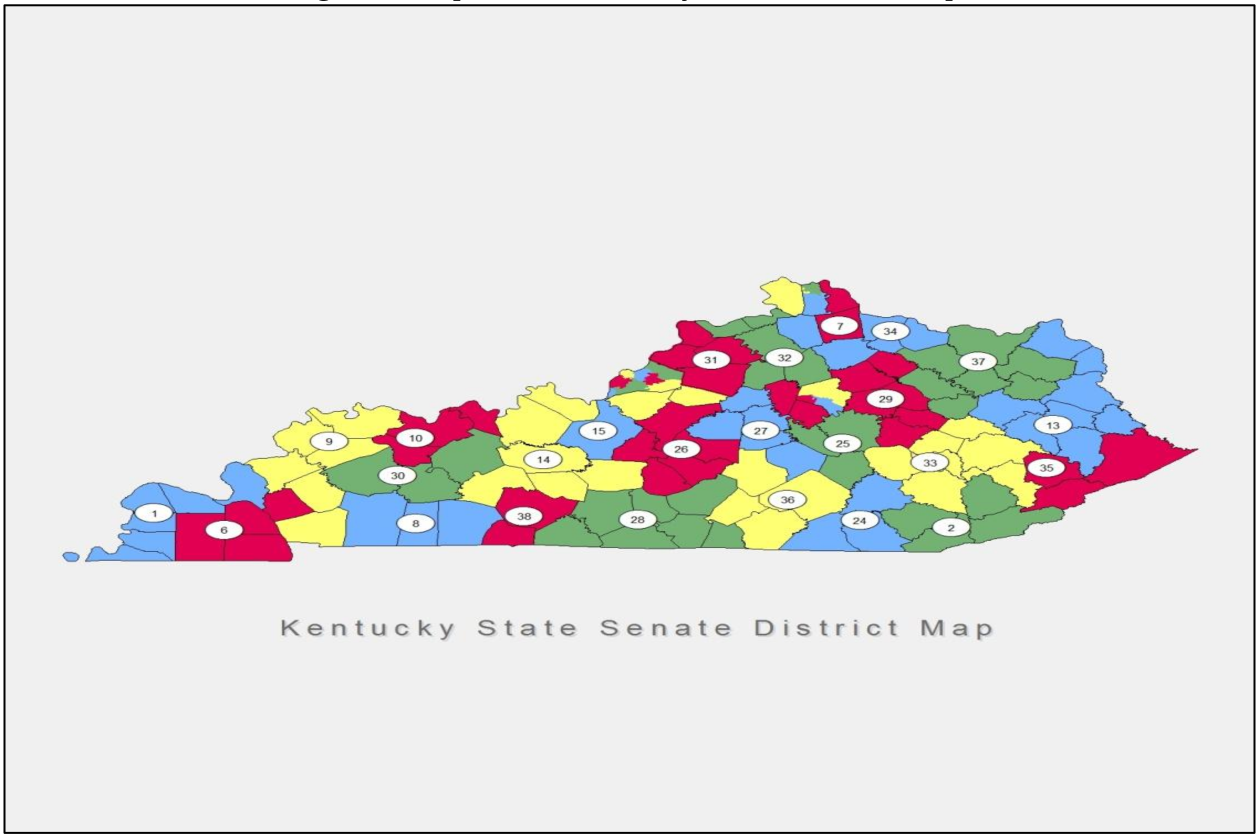
Figure 13: Urban area District Chosen in Blue



4.5 Optimization Results

Figure 14 shows the Kentucky Senate districting map created by the compact districting model developed in this thesis. Compared to the district maps of 1996 and 2002, shown in Figure 15 and 16, respectively, the model generated districts have much simpler shapes although they are not as circular as one might expect. In general, Kentucky senate districts are moderately compact (but see districts #5 and #25 in Figure 16).

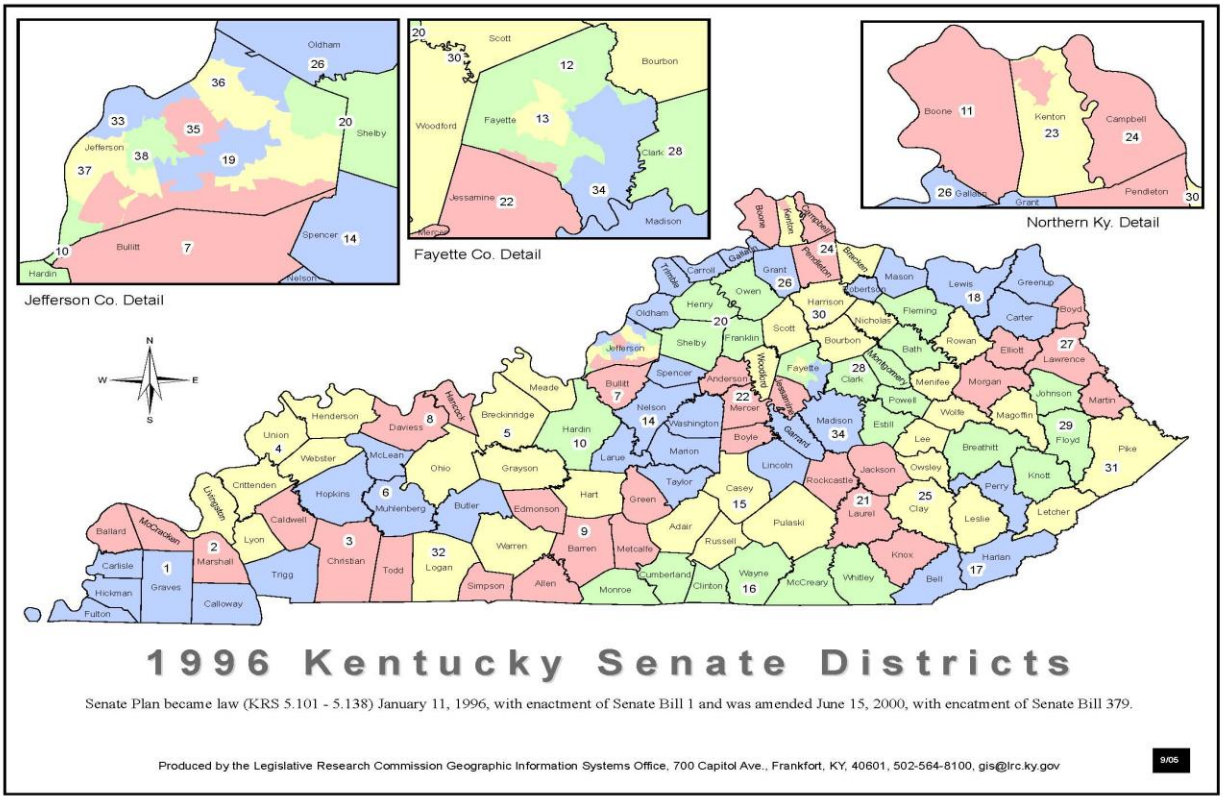
Figure 14: Optimized Kentucky Senate District Map



4.5-A. County Division

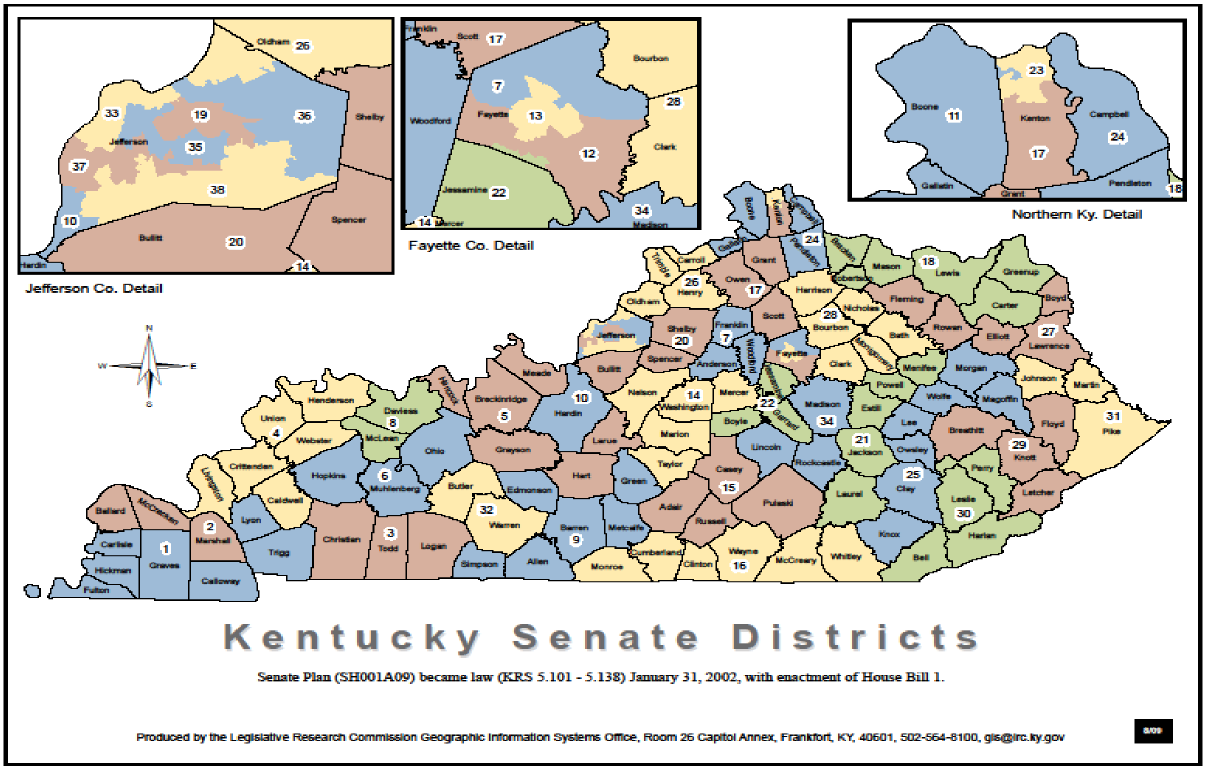
The total number of county divisions is the total number of districts that the divided counties belong. The Supreme Court held that the 1993 proposed state Senate districting plan that had 19 county divisions was unacceptable. An acceptable districting plan should contain as few county divisions as possible while still creating contiguous districts containing a population within 5% of the ideal district population. In 1996, a districting plan was accepted (shown in Figure 15). This plan contained 16 county divisions. The current Kentucky plan adopted in 2002 contains 15 divisions. In contrast, the districting plan created using the compact districting model presented here results in 14 county divisions, one less than the best plan that could be generated so far. Thus, in terms of county divisions, the model performed well and reduced the number of divisions by 26.3% over the rejected 1993 plan, by 12.5% over the plan accepted in 1996, and finally 6.7% over the plan accepted in 2002 (Figure 16).

Figure 15: Kentucky Senate District Map as of 1996[15]



[15] Kentucky Legislative Research Commission (1996)

Figure 16: Current Kentucky Senate District Map as of 2002[16]



4.5-B. Compactness Comparison

Another interest is how effective was the compact districting model at creating a compact districting plan. I will only make comparisons to the current plan in regards to compactness because spatial data is not available for the rejected proposed plan. As often stated in the literature, there is no perfect measure of compactness (Young, 1988). In the hope that where one measure lacks another measure makes up for it, I will use three different measures of compactness to attempt to attain a clear representation of differences in compactness between the district plan of the optimization model and the actual current district plan.

The first measure of compactness is population-weighted distance incorporated in the objective function of the model used in this thesis. It is a measure of the dispersion of units within a district. This measure is the only one that explicitly includes the locations and

[16] Kentucky Legislative Research Commision (2002)

magnitudes of the people along with the geographic shape of the district. Population weighted distance is defined as the total distance traveled as the crow flies for every person in Kentucky to reach the population weighted center of her district. In order to make a comparison to the actual district plan, I needed the distance from every census tract to the center of each district. I used an optimization process to attain the total population weighted distance for the actual Kentucky Senate district plan that would be comparable to the measures from the plan created in this thesis. I determined the center of each district by finding the census tract that minimized the total population weighted distance of the district. The final objective value is the population-weighted distance for that district. In the aggregate, the compact districting model increased compactness by reducing the total miles traveled by 7.3 million miles, which is a substantial improvement of 14.7% over the actual 2002 districting plan. Therefore, besides fewer county divisions, the optimal districting plan creates districts that are more compact than the actual current Kentucky Senate district plan. These findings may have important economic implications, namely candidates running for state senate will be able to reach the voters by traveling a significantly reduced total distance, which would cut their campaigning costs. More importantly, being closer to their constituents state senators would have stronger ties with them, can communicate more easily, and in the end will be able to represent and serve better.

Table 1: Average Population Weighted Distance Measures[17]

Actual Senate Districts	1.35 ± 0.83
Optimal Senate Districts	1.15 ± 0.71

The next measure used for comparison purposes is the Inverse Roeck test of compactness. This measure takes the area of the smallest circle that would fully encompass a district divided by the area of the district. A circle is the most compact shape using this measure.

[17] Measured in millions of miles.

The lower the value the more compact the district is. This measure takes a minimum value of one if the district is a perfect circle. The Inverse Roeck test is a measure of area density (Spann et al., 2007).

$$\text{Inverse Roeck} = \frac{\text{Area of Circle}}{\text{Area of District}} \tag{4.9}$$

Table 2: Average Inverse Roeck Measures

Actual Senate Districts	2.87 ± 0.81
Optimal Senate Districts	2.32 ± 0.55

The average Inverse Roeck score is lower for the optimal district plan with less variation implying that the optimal districting plan contains districts that are more compact with less variation in compactness among districts.

The final measure is called the Schwartzberg test for compactness. Using *Spann*'s transformation, the Schwartzberg compactness index is the perimeter of the district divided by the square root of 4π times the area. This measure assumes a circle is the most compact shape with more compact districts receiving a lower score that reaches a minimum value of one. The Schwartzberg test is a measure of overall skewness of the district (Spann et al., 2007). The score is increased when parts of the district jut out.

$$\text{Schwartzberg} = \frac{\text{Perimeter}}{\sqrt{4\pi \text{Area}}} \tag{4.10}$$

Table 3: Average Schwartzberg

Actual Senate Districts	2.11 ± 0.37
Optimal Senate Districts	1.76 ± 0.26

Although a different measure of compactness is used here when creating the optimal district configurations, the optimal districting plan is more compact than the actual districting map with respect to all three different compactness measures discussed above with less variation among districts. While the above results may suggest some correlation between the three compactness tests, this is just one instance and cannot be generalized. The main conclusion that can be drawn from the above comparison is that the Kentucky Senate districting plan created using the method proposed in this thesis is more compact in terms of the dispersion of people, the area density, and the skewness of district shapes. Compactness has not been an explicit criterion required by Kentucky's Constitution. However, a more compact district plan than the incumbent plan should be considered as a more preferable plan because of its spatial simplicity. Since, in addition, it divides counties less; it is a clearly dominating alternative. Therefore, we can safely conclude that the mathematical programming approach used here clearly outperforms the methods used to create the existing and previous districting plans.

SUMMARY:

This chapter applied the optimization method introduced in Chapter 3 to create a State Senate district plan for Kentucky. The model was adapted to insure a feasible solution is found using a reasonable amount of computing power and time. The resulting district plan improved on the previously proposed and actual Senate district plans in terms of both county divisions and compactness.

Chapter 5: Effects of Compactness

The previous chapter (specifically Section 4.5-B) makes it clear that the optimal districting plan generated by the optimization model contains districts that are more compact than Kentucky's current Senate district plan. This chapter aims to determine what effects compactness has on election results and the demographic composition of districts.

5.1 Effect on Election Results

In Chapter 2, I introduced common political districting criteria. In Chapter 3, I explained how to incorporate some of these criteria. I also stated that legislatures could choose multiple criteria to include in a district plan to meet their needs with the caveat that the inclusion of one criterion may worsen another. I will show here that the effects of altering a model and therefore district boundaries may also change election results.

I use 2004 and 2006 Kentucky State Senate election results to compare the outcomes holding all else equal except for the district boundaries. I compare the district plan created in this thesis to the actual district map.

In a given election year, half of the Senate seats are on the ballot. Therefore, to determine the statewide effects of a districting plan I use the results from last two elections. Socioeconomic characteristics are reported at census tract level. To obtain the socioeconomic characteristics of the actual and the model generated compact districts, I used two methods. For all districts that did not contain a portion of one of the three very populous counties, I used county level voting data and aggregated these results to the district level. For the districts that contain a portion of the three populous districts, I found the closest census tract to each voting precinct and appended the voting results to that census tract. Voting results from multiple precincts are added to each census tract, which are then aggregated to the district level. It is possible to have minor errors for census tract-precinct association generated this way if the precinct location for an individual is not within the census tract he/she resides.

Based on the results of past two elections, at the state level 48.6% of the votes cast were for Democrats, 50.1% for Republicans, and 1.3% for other candidates including independents. I made the claim that a 'fair' election outcome is when election results equal (as nearly as possible) voting results. A fair election outcome in this case would result in a Senate containing 18 Democrats, 19 Republicans, and 1 other Senator.

Table 4: 2004-2006 Kentucky Senate Voting Results

Party	Voting Results
Democrat	48.6%
Republican	50.1%
Other (including Independents)	1.3%

The election results given each district plan are displayed in Table 5. The actual election outcome included 17 Democratic Senators, 20 Republican Senators, and 1 Independent Senator. Using the same voting results only using the district boundaries created in the optimal district plan gives a substantially different outcome. Three more Democratic Senators would be elected according to the model generated districting plan. More than half of the Senate seats would be held by Democrats even though Democrats received less than half of the overall votes. Two less Republicans would be elected shifting the majority to Democrats. The Independent candidate loses his Senate seat according to the model results. The district election results for the actual district plan and the optimal district plan are displayed in Figure 17 and Figure 18 respectively. These results demonstrate how important drawing the district boundaries can be in terms of the election results. Even if the model presented here is not implemented in the actual districting process, these results can be highly valuable for educating the general public and draw their attention to the process.

Table 5: 2004-2006 Kentucky Senate Election Results

Party	Election Results Using Actual District Plan	Election Results using Optimal District Plan	'Fair' Election Results
Democrat	17	20	18
Republican	20	18	19
Other (including Independents)	1	0	1

Figure 17: Actual Election Results by District

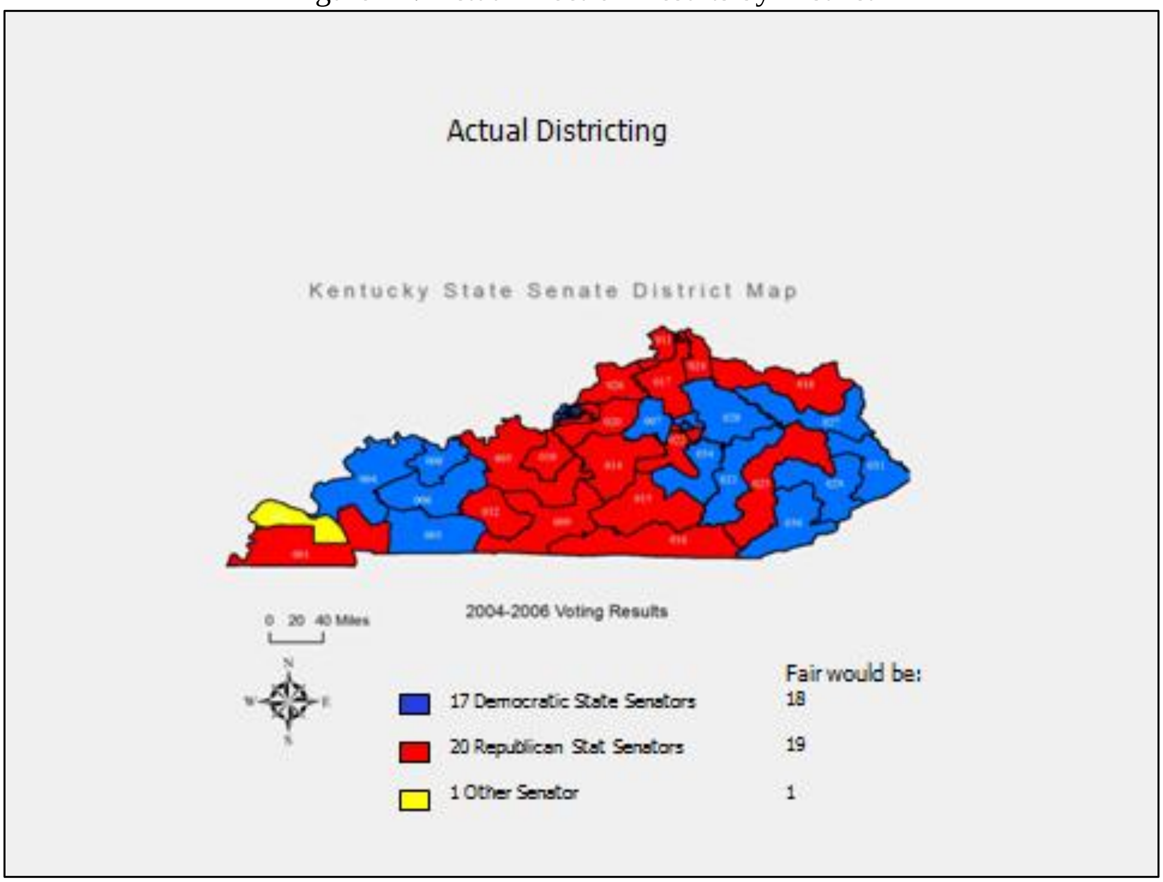
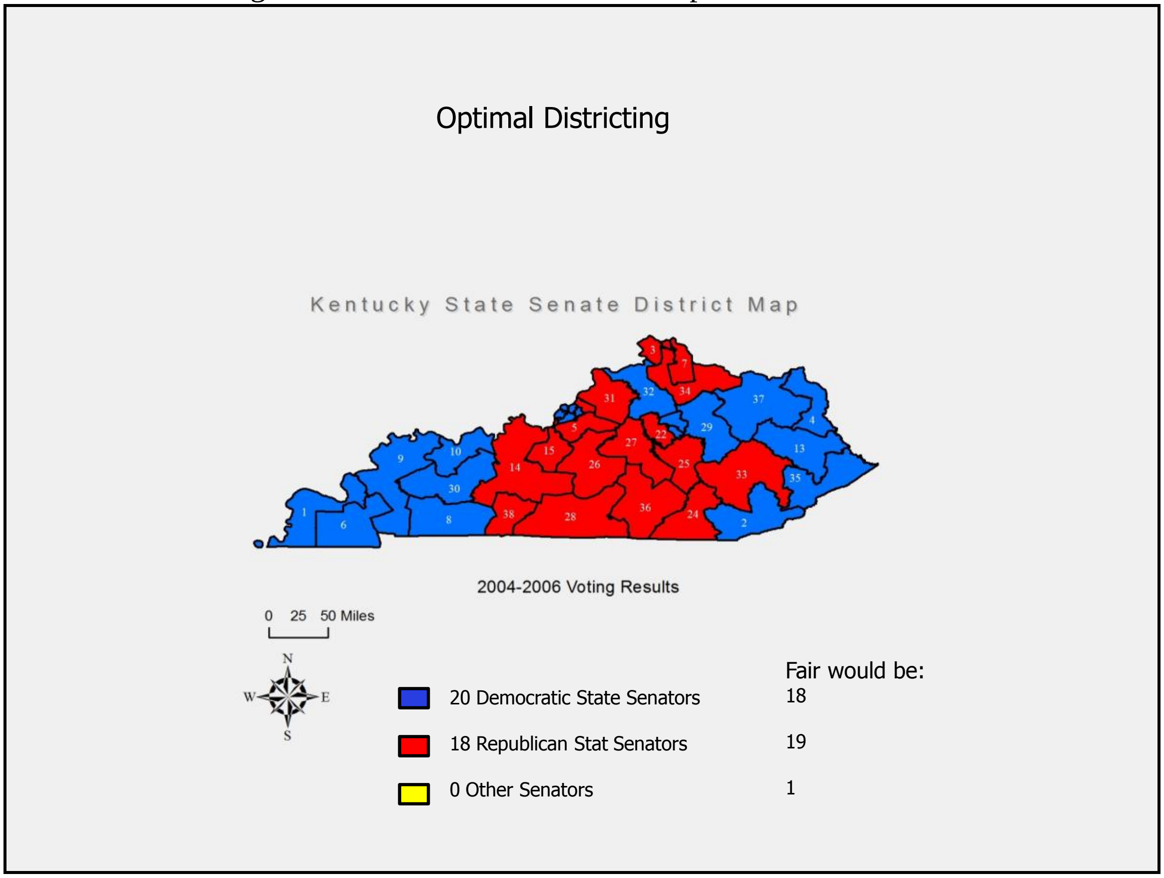


Figure 18: Election Results Given Optimal District Plan



Limitations

I made explicit and implicit assumptions that may significantly affect the results of this section. I assume redrawing district boundaries does not affect voting results. This assumption is necessary to make the comparison between election results from the two district plans. However, many voters put at least some weight into the candidate beyond the political party affiliation. This means that a voter may change her/his vote if s/he is placed in a different district with other candidates. I also assume that campaign strategy is unaffected by changing district boundaries. A different voting base is likely to necessitate a different campaign strategy for all parties. Finally, I use voting results for one election cycle. It would be preferable to attain the voting results over an extended period so the data would be more reliable.

5.2 Effect of Compactness on District Demographics

Is compactness associated with the demographic composition of a district? It is possible that a district comprised of similar types of people would be easier to represent. The representative can follow the unified voice of the constituents and would be more passionate about working for their common interests. Alternatively, diversity at the district level may not be a bad outcome. Districts with very few homogeneous groups, such as two groups with opposite and strong political affiliations or too distinct economic characteristics (wealthy and very poor), can be so polarized on important issues that compromises may never occur. I take no strong stance on what is the preferable outcome; rather I just want to measure uniformity of the compact districts configured by the optimization model vis-à-vis uniformity of the actual state senate districts in Kentucky. The following sections focuses on what relationship, if any, compactness of a district has on the grouping of similar types of people. I will determine if there is any effect on grouping people of a similar race, educational background, industry one works for, occupation type, or income level. I look at individual district level and the aggregate level, comparing between the actual district plan and the optimal district plan.

5.2-A. Compactness Measures

For the purpose stated above, I will use the three measures of compactness introduced in Chapter 4. Each of the measures is a proxy for different forms of compactness. The population-weighted total distance from the center of a district is an absolute measure of the dispersion of people within that district. As with any absolute measure of compactness, a negative bias towards geographically large districts is implicit. The Inverse Roeck and the Schwartzberg are both relative measures that evaluate the compactness of the shape of the district without regard to the geographic size of the district. Each is individually used as independent variables in separate regressions in order to determine if any measure of compactness shows a different relationship to the demographic dispersion measures.

5.2-B. Socioeconomic Data

I gathered demographic data for each census tract in Kentucky from the summary three files of the 2000 decennial Census. The focus will be on five demographic and socioeconomic grouping of district inhabitants. It might be interesting to know the relationship that compactness has on the grouping people of the same race together. I categorized the population into three race groups: White, Black, and Other. Of the population, 90.1% is White in Kentucky, 7.3% is Black, with 2.6% falling into the Other category.

Table 6: Percent of Kentucky Population Belonging to Each Race

Race	**Percent**
White	90.1%
Black	7.3%
Other	2.6%

The next two categories are closely related. I will determine if, all else equal, compact districts are more likely to contain people that work in the same general industry or in the same type of job. Four types of jobs are identified for this purpose: white collar, blue collar, service, and agriculture. Of the working population 30.7% are involved in blue collar jobs including construction, maintenance, repair, production, and transportation jobs. About 53% of all workers in Kentucky are categorized as white collar, 14.3% are categorized as people working in service jobs and 2% as having agricultural (farm related) jobs.

Table 7: Distribution of the Working Population by Occupation

Occupation Type	Percent
White Collar	53.0%
Blue Collar	30.7%
Service	14.3%
Agriculture	2.0%

I will finally relate the compactness of a district to grouping of people with similar educational backgrounds and income levels. The Census lists the maximum level of education attained for everyone over the age of 25. I used seven educational attainment levels: about 4% of the population over age 25 had completed 6^{th} grade or less; 7.7% had only made it to 7^{th} or 8^{th} grade; 14.2% completed some high school but not graduated; 33.6% had a high school diploma but no college; 23.5% had attended at least one year of college but not graduated; 10.2% had a degree from a four year institution; and finally 6.7% had gone beyond a four-year college degree. For income levels, I separated the total income per capita range for Kentucky into quintiles. Table 9 displays the distribution of Kentucky residents among five income ranges .Of all the Kentucky residents, 34.6% of the people had an income per capita of less than $14,918; 88.2% of Kentucky had an income per capita of less than $24,602; 96.8% of Kentucky of less than $34,287; and 99.2% of Kentucky had an income per capita of less than $43,971.

Table 8: Distribution of Population over 25 by Highest Educational Level Attained

Highest Educational Attainment Level	Percent
6^{th} Grade or Less	4.0%
7^{th} or 8^{th} Grade	7.7%
Some High School	14.2%
Completed High School	33.6%
Some College	23.5%
Completed Four-Year College	10.2%
Beyond Bachelor's Degree	6.7%

Table 9: Distribution of Kentucky Population by Income Level

Income Per Capita Categories	Percent
Less than $14,918	34.6%
$14,919 - $24,602	53.6%
$24,603 - $34,287	8.6%
$34,288 - $43,971	2.4%
Above $43,971	0.8%

5.2-C. Statistical Procedures

The goal of this section is to relate the compactness of a district to homogeneity of the people within that district. For this, I use the Shannon Information Index (H) as a proxy for the degree of dispersion among the different classes within a demographic set. Shannon index is defined as follows:

$$H = -\frac{\sum_{i=1}^{S} p_i \ln(p_i)}{\ln(S)} \tag{5.1}$$

where p_i is the percent of the total district population of the ith class of a demographic set and S is the total number of classes within the demographic set. The index takes a minimum value of zero when the entire district population is comprised of one class and a maximum value of 1.0 when the population is equally divided between the different classes. A value closer to zero implies that the district is more homogeneous in terms of the demographic criteria of interest (similar people are grouped together, i.e. the entire district contains one group of people with the same demographic characteristic), while higher values imply that the district is more heterogeneous (there is a very diverse assortment of people within the district).

The Shannon Index measures are calculated for individual districts and regressed on each of the compactness measures using Ordinary Least Squares to determine if there is any relationship between compactness and the similarity of people within a district. The regressions are done on the district plan created here by using the optimization model (labeled 'optimal') and on the current senate district plan (labeled ‘actual’). It should be noted that the results obtained here have some limitations because of the small sample size, namely thirty-eight districts that comprise the Kentucky’s Senate district plan. In addition, even if there is or is not a significant relationship between compactness and socio-economic characteristics of political districts in Kentucky, the results cannot be generalized to all states since the case of Kentucky is just one instance and may be an outlier if the analysis was carried out for all political districts in all the states in US. The differences between the composition of Kentucky residents from the composition of residents in other states (which is not looked at here) and the variance at the state level can influence the results.

5.2-D. Results

Table 10 displays the results of the OLS regressions. A positive coefficient value indicates that as the district becomes more compact, it is increasingly likely that the district contains people from the same demographic group. Each square in the table contains the coefficient and standard deviation using the optimal and actual district plans for a particular compactness measure and demographic set.

The Inverse Roeck and the Schwartzberg measures of compactness are weakly related or not related at all to the grouping of people within a district. They indicate inconsistent results when applied to the actual district plan versus the optimal plan. For instance, the coefficients on the Schwartzberg measure of compactness in relation to educational attainment is 0.0174 for the optimal plan indicating compactness is associated with districts that contain people from the same educational background. Using the coefficient on Schwartzberg measure of compactness (-0.0037) on the actual plan implies that compactness defined by the Schwartzberg Index is negatively associated with districts containing more people from the same educational background. None of the results for either of these two relative measures of compactness is statistically significant even if relaxed to the 15% level. The p-values are all greater than 0.15.

In the following discussion, the 'actual' district map refers to the Kentucky State senate district map that was in effect in 2002. The 'optimal' district map refers to the district map that is generated by using the optimization model developed for determining a compact Kentucky senate districting plan (i.e. the model described in Chapter 4).

Schwartzberg

The Schwartzberg compactness index is the perimeter of the district divided by the square root of 4π times the area.

$$\text{Schwartzberg} = \frac{\text{Perimeter}}{\sqrt{4\pi \text{Area}}} \tag{5.2}$$

To explain what (5.2) measures, think of a district as an octopus. Increasing the Schwartzberg compactness of a district reduces the number and relative size of the tentacles. As a district becomes more compact, the tentacles disappear eventually reaching an optimally compact circular district.

The results show that the Schwartzberg compactness measure is inconclusive about the relationship between compactness and relative uniformity of the districts in terms of their demographic and socio-economic characteristics. This is shown by the lack of asterisks (***) in Table 10, where a *** indicates a p-value less than 1% and ** indicates a p-value less than or equal to 5%. Asterisks imply that the coefficient is statistically significant (and insignificant otherwise). In the case of education characteristics, for instance, applying Schwartzberg compactness to the actual district map shows that increasing Schwartzberg compactness is associated with an increase in the number of similarly educated people living within a district (indicated by the positive coefficient 0.0174). However, this relationship is not statistically significant, indicated by a p-value of 0.43. The opposite result is observed when this compactness measure is applied to the optimal districting plan, where the coefficient is negative (-0.0037) but this relationship was again statistically insignificant with a p-value of 0.31.

Similar results are observed regarding the relationship between compactness and all other demographic/socioeconomic categories except for the income per capita category. Schwartzberg compactness is associated with less uniform income distribution for both districting plans, namely the distribution of people in a given district among the different income levels becomes less uniform (indicated by the negative coefficients -0.2591for the actual district map and -0.0864 for the district map generated by the model) as the district becomes more compact.

Inverse Roeck

The Inverse Roeck test of compactness takes the area of the smallest circle that would fully encompass a district divided by the area of the district.

$$\text{Inverse Roeck} = \frac{\text{Area of Circle}}{\text{Area of District}} \tag{5.3}$$

An analogy for compactness in terms of Inverse Roeck, is to imagine a district as an irregularly shaped piece of construction paper placed inside of the smallest hula-hoop that can fully encompass the piece of paper. In the case of imperfect compactness, there would be open spaces between the piece of paper and the edge of the hula-hoop. The district would be more compact when the open space within the hula-hoop becomes smaller. It would be optimally compact when the piece of construction paper and the hula-hoop are of the exact same size and shape, in which case the district is perfectly circular.

Similar issues arise with the Inverse Roeck measure of compactness in describing the grouping of people in terms of income per capita and occupation. As in the case of Schwartzberg compactness measure, increasing the Inverse Roeck compactness of a district leads to inconsistent (opposite) effects in the actual and optimal district plans. Specifically, in the case of optimal district plan an increased compactness increases uniformity in the income distribution (shown by the positive coefficient 0.0008) and decreases uniformity in the distribution of people among occupation groups (indicated by the negative coefficient -0.0051). The situation is opposite in the case of actual district plan (where the respective coefficients are -0.0768 and 0.0021). Inverse Roeck compactness is associated positively with grouping people from the same industry within the same district. This result is consistent for both districting plans. On the other hand, increasing compactness is associated negatively with the distribution of people from the same educational background and race. In all cases, however, both the positive and negative coefficients are statistically insignificant.

Population-Weighted Total Distance

The population weighted total distance as a measure of compactness yields results that are more consistent with intuition. In addition, the findings are robust among the actual and optimal district plans and all coefficients are statistically significant at the 5% level while most at a tighter 1% level. Three asterisks (***) next to a coefficient indicates significance at the 1% level. Two asterisks (**) next to a coefficient indicates significance at the 5% level. The population weighted distance compactness measure is an absolute measure of how far people are away from each other in a district. The results indicate that the closer people are (spatially) within a district the more likely they are to be from a similar educational background or work in a similar type of job. On the other hand, such districts are associated with greater dispersion of people in terms of income levels, the industry they work for, and their race.

The above findings may have important implications depending on which one of these characteristics influence election results most. For instance, if voting behavior were largely determined by income level, then a compact district would have mixed election results, i.e., the distribution of votes among parties or candidates would be more dispersed, since the income distribution would be more dispersed in such districts. However, as noted earlier, these results cannot be generalized since the case study considered here is just one instance, more empirical evidence is needed to be able to make a conclusion. In addition, although there are some exceptions the Kentucky state senate districts are in general compact and nicely shaped, gerrymandering is not a serious and widespread phenomenon. Performing the analysis presented here for states that have seriously gerrymandered districts, such as congressional districts in Illinois, would shed more light on the relationship between compactness and socio-economic/demographic characteristics of political districts. This topic requires further research in this area.

Table 10: Effects of Compactness on Homogeneity of Districts

Characteristic	District Plan	Inverse Roeck	Schwartzberg	PWTD[1]
Education	Optimal	-0.0090 (0.0101)	0.0174 (0.0208)	0.0169** (0.0073)
	Actual	-0.0051 (0.0064)	-0.0037 (0.0140)	0.0156*** (0.0057)
Income	Optimal	0.0008 (0.0554)	-0.2591 (0.1061)	-0.1441*** (0.0350)
	Actual	-0.0768 (0.0343)	-0.0864 (0.0779)	-0.1220*** (0.0292)
Occupation	Optimal	-0.0051 (0.0229)	0.0782 (0.0457)	0.0744*** (0.0124)
	Actual	0.0021 (0.0152)	-0.0186 (0.0328)	0.0597*** (0.0110)
Industry	Optimal	0.0049 (0.0049)	-0.0033 (0.0103)	-0.0108*** (0.0034)
	Actual	0.0023 (0.0036)	0.0080 (0.0078)	-0.0088*** (0.0032)
Race	Optimal	-0.0761 (0.0490)	-0.2501 (0.0961)	-0.1299*** (0.0322)
	Actual	-0.0323 (0.0341)	0.0155 (0.0748)	-0.1221*** (0.0268)

1 Population-Weighted Total Distance
***1% Significance level
**5% Significance level

Plan Level Comparison

I compare the actual plan to the optimal plan to determine if a district plan with compactness considerations contains a different dispersion of people. The results are displayed in Table 11. Districts on average are more likely to contain people from the same educational background, income level, and occupation. This is shown by a lower average Shannon Index for these characteristics using the optimal plan. The optimal plan is less likely on average to contain people working in the same industry or the same race. This is shown by higher Shannon Index average values for the optimal plan. It is surprising that the results for the occupation and industry characteristics are not consistent. The result for race follows the need that majority-minority districts be less compact. Using the Student's T-test for differences in means, I determined that the average Shannon Index values between the actual and optimal plan are not statistically different from one another for any characteristic.

Table 11: Average Shannon Index by Plan and Characteristic

District Plan	Education	Income	Occupation	Industry	Race
Actual	0.8582	0.4260	0.7490	0.9009	0.2917
Optimal	0.8563	0.4245	0.7484	0.9013	0.2932

SUMMARY:

I used voting data for a full Senate election cycle to show that redrawing district boundaries can impact election results. The only compactness measure that had statistically significant results in regards to any other demographic composition of a district was the population -weighted distance measure, which is used in the model developed in this thesis to promote compactness and contiguity. This implies that the absolute distance between people in a district is more informative as an indicator for the similarity of people, than the relative measures of compactness. The Inverse Roeck and the Schwartzberg are relative measures that only indicate how similar the district shape is to a circle regardless of the size of the district. The relative shape of the district has no statistically significant association to the grouping of similar people within a district. These results are defensible. They indicate that there is no relationship between demographic and socioeconomic characteristics and how close a district is to a circle in terms of shape. The proximity of people to one another is more informative in regards to how similar people are within a district.

Chapter 6: Summary, Conclusions, and Further Research Issues

In the upcoming year, following the 2010 census, political districting will be on the minds of many people. All states will reconfigure their political district boundaries, including both congressional and state legislative districts. This thesis addressed this timely problem and introduced a mathematical programming modeling method for generating contiguous districts with almost equal population. Besides these two basic criteria different states have other criteria employed in political districting, such as compactness, community integrity, minority representation, etc. This chapter gives a short review of the modeling method used in the research, summarizes the major findings of the model used, and presents a discussion of the model's limitations. Finally, further research issues will be discussed.

6.1 An Overview of the Thesis

Population results from the census may necessitate redistribution of representative seats. This reapportionment requires district boundaries be redrawn to accommodate the population changes. This is known as political redistricting.

This study introduced a methodology to draw district boundaries where the main criterion, namely population equity, is incorporated as a constraint while the second main criterion, namely spatial contiguity, is achieved in an indirect way. The districting approach proposed in this thesis is generic and may serve as a framework for an unbiased method of redistricting for any state and for any type of political districting, including congressional or state legislative districts. This method is capable of using real data to create a district plan in a large state that creates districts that are compact, contiguous, and meet the requirements of the state.

An optimization model, specifically a linear integer program, was introduced for aggregation of base units that form a district. The model is a variation of a generic formulation, known as the *p-center formulation*, adapted to the particular problem addressed here. This type of ‘clustering’ problem involves distances between a number of central locations and the spatial units assigned to each center (thus the name ‘p-center’) among other considerations and objectives. The model developed here chooses units to be district centers and attaches the closest units to those centers in such a way that the total distance between the district center and the units assigned to that center, summed across all districts, is minimized. This is done by defining yes-no type decision variables (‘yes’ means assign, ‘no’ means do not assign a particular unit to a particular district center) represented by binary (0-1) variables in the model. The model considers a number of potential centers for districts and selects a subset of them that actually serve as district centers together with the unit assignments. The district centers have no ‘real world role’, rather their artificial role is to make the mathematical model work and allow the aggregation of base units according to the specified criteria. The total distance can be viewed as a measure of compactness, but this is actually a secondary purpose. The main purpose of this model formulation is to create contiguous districts, which is a required property in political districting. Minimization of the sum of distances to district centers generally selects units that are adjacent to their centers or to each other, thus forming spatially contiguous clusters of base units.

6.2 Major Findings and Conclusions

The compact districting model developed in Chapter 3 is applied to the Kentucky State Senate districting problem. The model was able to produce a district plan that was significantly more compact than the current plan and also reduced the number of county divisions, which is considered by the State of Kentucky Supreme Court as an ‘indispensable’ criterion in political districting. In general, the idea behind compactness is to group people with similar socio-economic and demographic characteristics, which is thought to lead to facilitate a better representation of the constituents by the district representative. Although this is a common belief, a quantitative analysis has been lacking in the literature. This thesis provided a unique opportunity to do that. I analyzed the relationship between compactness and the grouping of people in terms of demographic and socio-economic characteristics using the census data at tract level, the actual districting map implemented by Kentucky in the past two elections and the model generated districting map. The Schwartzberg and Inverse Roeck measures of compactness are seen to be poor indicators of how people are grouped. The population weighted distance compactness measure, on the other hand, showed promising results and indicated that the objectives of compactness, namely grouping people with similar characteristics, can be achieved to some extent by maximizing compactness (minimizing the total population-weighted distance). How close people are to each other has more to do with how similar they are than the relative shape of the district they live in.

Two interesting findings arise from the results in terms of the population weighted distance compactness measure. The compact districts are more likely to have residents from the same educational background and have a similar type of job. This is an important finding since people from similar educational backgrounds and working in the same type of jobs (métiers) will likely have similar political interests. A compact district may be a preferable outcome from the perspective of the representative. S/he will be able to represent more easily a group of people with similar interests and from similar educational backgrounds and be a unified and more passionate voice of most of her/his constituents than too many voices all yelling for something different.

The next result relates to the ethnic\racial composition of districts and is consistent with the need to create majority-minority districts that are less compact. In the particular case studied here, the results show that the proximity of people within a district is negatively associated with the likelihood that they are from the same race, as shown by the negative coefficient in the estimated relationship between compactness and racial homogeneity. Therefore, in order to create a majority-minority district, less compact districts need to be formed. This is in fact the very reason for highly gerrymandered districts when minority districts are to be generated. Unless sufficiently large minority populations are concentrated in certain geographical areas, which might be the case in some circumstances but may not always be the case, non-compact and gerrymandered districts will be needed to put together several pockets of minority populations in one district. These results pertain only to districts in Kentucky and should not be generalized to all states. Further research is needed to test the robustness of the results for state level differences.

6.2-A. Limitations

The computational complexity (difficulty in getting numerical solution) of the model presented here increases as the number of districts and base units increase since this requires a larger integer programming model. In general, integer programming models with a larger number of binary variables are harder to solve. Even if they are solvable, more processing (solution) times may be required. Assuming that the generic model can be applied to any political districting problem without any problems whatsoever may be a misconception. Computational problems may arise in any application, but with skillful data processing techniques it is possible to overcome such difficulties. In the particular application presented here, in order to reduce the model size (in particular the number of binary decision variables) the model was fine-tuned by aggregating a large number of census tracts with a relatively small population into larger ‘indivisible’ units and in many cases aggregating them to the county level.

Limiting the number of potential centers also reduces the size of the model. The set of potential district centers (which are actually census tracts or aggregated tracts) was specified up front by spreading them out almost uniformly across the state. Such data processing steps may be necessary in most empirical applications, but this is not considered as a serious deficiency since these steps can be performed in a straight forward manner by developing a programming component that generates the set of aggregated tracts and specifying potential district centers as it is done in this study.

Using computers and optimization techniques does not result in just one map that eliminates human interaction. The model should be thought of as a tool that can produce many districting maps adapted to meet the concerns and criteria of the legislature. It is not a replacement of legislators, instead it facilitates the process and allows legislators to choose from many different possibilities. The model makes the process easier and hopefully creates better alternatives than districting maps that can be created by manual or judgmental approaches.

6.3 Further Research

The next logical step in the progression of this research is to apply the general compact districting model to other states to see if it is capable of producing acceptable and improved district plans compared to the actual plans. Every state has a specific set of criteria that a district plan must follow. Adapting the method used in this thesis to varying numbers of base units and criteria can be a challenging task in some cases, particularly when minority districts are required. Technically, it is possible to add constraints to generate a desired number of minority districts, but contiguity of those districts is a difficult issue. The model used here would not in general guarantee contiguity in such cases. A possible method is to use a 'column generation' method as in Nemhauser and Wolsey (1988) and Mehrotra et al. (1998).One can generate a number of alternative minority districts, without consideration of compactness, using the approaches introduced by Önal and Briers (2005) and Cerdeira et al. (2005) that guarantee spatial contiguity. Those district configurations can then be imported into the model used here as possible alternatives. The model would then select the most appropriate minority district configuration(s) while designing the remaining districts with consideration of compactness. It should be noted, however, that this is not going to be a simple procedure. Practical/computational problems may arise at any step and modeling/computation skills of the analyst would be crucial. Thus, neither this thesis nor any other study can claim that the districting problem is solved once and for all. Every districting problem may be a challenge and may require innovative approaches/techniques to address the specific problem at hand.

There is already a plan to apply the model introduced in this thesis to the state senate and legislative districting for the state of Illinois. Illinois poses a potentially more difficult problem than Kentucky because of the large number of spatial units (census tracts), approximately 3000 more than Kentucky. Illinois also has a population exceeding 12 million people with nearly half living in Cook County. The districts to be generated in that area have to be geographically small and very fine tuned spatial analysis would be needed. This is a challenging research project and the computational difficulties of applying this model to Illinois have yet to be seen.

One of the simplifications that made the solution tractable was selecting a sub-set of units to be considered as possible centers. In this thesis, I choose 100 units approximately equally spaced throughout Kentucky. It would be preferable to have the model endogenously choose the sub-set of possible centers based on population density, spatial, geographic, and possibly other characteristics instead of choosing units a priori. The political districting model could then be a two-staged model that first selects the best selection of possible centers, then proceeds to create an optimal district plan.

Appendix A: GAMS Code

I used GAMS to code and CPLEX to solve the optimization model introduced in this thesis. This appendix contains the GAMS code in its entirety. Asterisks indicate notes that describe lines or blocks of code directly below the asterisked line.

```
        $Title ' KY Model County Division'

        SET   UNITS spatial decision units in Kentucky /SC1*SC994/
              COLS columns /POP,County, Cent, /
              COUNTIES county names
/ADAIR,ALLEN,ANDERSON,BALLARD,BARREN,BATH,
                    BELL,BOONE,BOURBON,BOYD,BOYLE,BRACKEN,
                    BREATHITT,BRECKINRIDGE,BULLITT,BUTLER,CALDWELL,
                    CALLOWAY,CAMPBELL,CARLISLE,CARROLL,CARTER,
                    CASEY,CHRISTIAN,CLARK,CLAY,CLINTON,CRITTENDEN,
                    CUMBERLAND,DAVIESS,EDMONSON,ELLIOTT,ESTILL,
                    FAYETTE,FLEMING,FLOYD,FRANKLIN,FULTON,GALLATIN,
                    GARRARD,GRANT,GRAVES,GRAYSON,GREEN,GREENUP,
                    HANCOCK,HARDIN,HARLAN,HARRISON,HART,HENDERSON,
                     HENRY,HICKMAN,HOPKINS,JACKSON,JEFFERSON,JESSAMINE,
                    JOHNSON,KENTON,KNOTT,KNOX,LARUE,LAUREL,
                    LAWRENCE,LEE,LESLIE,LETCHER,LEWIS,LINCOLN,
                    LIVINGSTON,LOGAN,LYON,MADISON,MAGOFFIN,MARION,
                    MARSHALL,MARTIN,MASON,MCCRACKEN,MCCREARY,MCLEAN,
                    MEADE,MENIFEE,MERCER,METCALFE,MONROE,
                    MONTGOMERY,MORGAN,MUHLENBERG,NELSON,
                    NICHOLAS,OHIO,OLDHAM,OWEN,OWSLEY,PENDLETON,
                    PERRY,PIKE,POWELL,PULASKI,ROBERTSON,ROCKCASTLE,
                    ROWAN,RUSSELL,SCOTT,SHELBY,SIMPSON,SPENCER,
                    TAYLOR,TODD,TRIGG,TRIMBLE,UNION,WARREN,
                    WASHINGTON,WAYNE,WEBSTER,WHITLEY,WOLFE,WOODFORD/

        ALIAS (UNIT,UNITS);

        *Pulls this table from a csv file in excel
        TABLE UNITINFO(UNITS,COLS)  population_which county_and whether its a center
for each unit
        $ondelim
        $INCLUDE C:\Documents and Settings\labuser\Desktop\tractpopulation.csv
        $offdelim

        SCALARS maxpop population of the largest unit
              minpop population of the smallest unit
```

```
        avgpop average population of all units
        KYPop total population of Kentucky;

        KYPOP=Sum(UNITS,UNITINFO(UNITS,'POP'));
        maxpop=smax(UNITS,UNITINFO(UNITS,'POP'));
        minpop=smin(UNITS,UNITINFO(UNITS,'POP'));
        avgpop=KYPOP/Card(UNITS);

PARAMETER POPULATION(UNIT) population of each tract
          Countynum(COUNTIES) county number;

          POPULATION(UNIT)=  UNITINFO(UNIT,'POP');
          Countynum(UNIT)=  UNITINFO(UNIT,'County')

*Units in the Center set if in the unitinfo table it has a value other than 0
SET  CENTER(UNITS) units that may serve as district centers    ;
CENTER(UNITS)=NO;
CENTER(UNITS)$UNITINFO(UNITS,'Cent')=YES;


*Pulls distance table from tractdistanceround csv file
TABLE DIST(UNIT, UNITS) distances between units
$ONDELIM
$INCLUDE C:\Documents and Settings\labuser\Desktop\tractdistanceround.csv
$OFFDELIM

SCALAR mindistance, index1,index2,index3, counter;

PARAMETER  ORIGUNITINFO(UNITS,COLS)  original unit info
           ORIGDIST(UNIT, UNITS)    original unit distances;

           ORIGUNITINFO(UNITS,COLS)=UNITINFO(UNITS,COLS);
           ORIGDIST(UNIT, UNITS)= DIST(UNIT, UNITS);

**LOOP line 1- for each unit(left row headers of dist table) index1 is the
*          ordinal # of that unit, counter set to zero, to calculate
*          mindistance look at every distance in that row and find the min
*          other than dist to itself
**LOOP line 2- loop through each distance in that row that = mindistance, make
*          index2 equal to that column headings ordinal number of that UNITS
**LOOP line 3- loop through that same row find any other dist = to mindistance
*          but not the same entry found in the line above
**LOOP line 4- when another distance = to mindist is found, add 1 to counter,
*          make that column # = index3, and display all
**LOOP line 5- change the distance entry slightly to make it slightly larger
**LOOP line 6- change the entry so the distance from location a to location b
```

```
*           is the same as the distance from the location b to location a

Loop(UNIT, index1=ord(UNIT); counter=0; mindistance=smin(UNITS$(Ord(UNIT) ne
Ord(UNITS)),DIST(UNIT, UNITS) );
        LOOP(UNITS$(DIST(UNIT, UNITS) eq mindistance ),index2=ord(UNITS));
         LOOP(UNITS$(DIST(UNIT, UNITS) eq mindistance and Ord(UNITS) ne
        index2) ,
        counter=counter+1; index3=ord(UNITS); display index1,index2,counter, index3,
mindistance;
DIST(UNIT, UNITS)$(Ord(UNITS) eq Index3) = DIST(UNIT, UNITS)+counter*0.001;
        DIST(UNITS, UNIT)=DIST(UNIT, UNITS);));

SCALAR TOTALDISTRICT total number of state senate districts /38/
     GROUP max # of units that can possibly go into a district /50/
     LIMIT percent deviation above or below the average population /0.05/
     AVGDISTPOP average district population;

          AvgDistPOP=KYPOP/TOTALDISTRICT   ;

PARAMETER Discard(UNIT) units to be discarded after aggregation
       Aggregate(UNIT,UNITS) 1 if units is aggregated to unit zero otherwise;



SCALAR MinDist determines nearest adjacent unit during aggregation of units /0/
     DiscardPop   population discarded
     AgrPopBefore agr population before
     AgrPopAfter   agr population after;

SET AGRUNIT(UNIT) aggregated units excluding units with less than 3000 pop;
parameter UnitInd(unit) assign the ordinal # to each unit before aggregation;
       UnitInd(unit)=Ord(unit);

SCALAR Discarded  unit discarded
     Aggregated aggregated unit;

**LOOP line 1- loop through each unit with less than 3000 pop.
**LOOP line 2 - the discard parameter is 1 one for that unit and discardpop is
*            the population of that unit
**LOOP line 3&4- find mindist in that row only looking at units in the same
*             county and can't be dist. to itself or it can't be a unit that
*             is already discarded
**LOOP line 5- now loop through each UNITS distance in that row where they are
*           in the same county and its not a discarded unit
**LOOP line 6- If the distance from UNITS to UNIT = mindist, then UNIT is
*           aggregated to UNITS and discarded= ordinal # of UNIT to be
```

```
*           discarded and aggregated= ordinal # of the hub UNITS that UNIT
*           will be aggregated to
**LOOP line 7- agrpopbefore= population of hub tract before aggregation
**LOOP line 8- population of discarded UNIT is added to hub UNITS
**LOOP line 9- agrpopafter = population of hub UNITS after aggregation

LOOP(UNIT$(UNITINFO(UNIT,'POP') LT 3000),
      Discard(UNIT)=1; DiscardPop= UNITINFO(UNIT,'POP');

  MinDist=SMIN(UNITS$(UNITINFO(UNITS,'COUNTY') eq
UNITINFO(UNIT,'COUNTY') and
   UnitInd(UNITS) ne UnitInd(UNIT) and Discard(UNITS) eq 0), DIST(UNITS,UNIT));
     LOOP(UNITS$(UNITINFO(UNITS,'COUNTY') eq UNITINFO(UNIT,'COUNTY')
and
       Discard(UNITS) eq 0 ),
        IF(DIST(UNITS,UNIT) eq MinDist,
           Aggregate(UNITS,UNIT)=1;
           discarded=ord(UNIT);
           Aggregated=UnitInd(UNITS);
             AgrPopBefore= UNITINFO(UNITS,'POP');

UNITINFO(UNITS,'POP')=UNITINFO(UNITS,'POP')+UNITINFO(UNIT,'POP');
           AgrPopAfter= UNITINFO(UNITS,'POP');
*   display discarded,Aggregated,DiscardPop,AgrPopBefore, AgrPopAfter;
) );};

*a UNIT is an agrunit unless it is aggregated to another tract
   AGRUNIT(UNIT)=YES; AGRUNIT(UNIT)$(Sum(UNITS,Aggregate(UNITS,UNIT))
gt 0)=NO;

*any UNIT that is an agrunit is not discarded
  Discard(AGRUNIT)=0;

SET discardunit(UNIT)no UNIT is part of this set unless it has been discarded in the
above loop;
     discardunit(UNIT)=NO;discardunit(UNIT)$Discard(UNIT)=YES;  ;
scalar totaggregate /0/
      totdiscarded /0/
      both;
      totaggregate=sum(AGRUNIT,1);
      totdiscarded=sum(UNIT$Discard(UNIT),1);
      loop(agrunit$Discard(AGRUNIT),both=UnitInd(AGRUNIT);)

**LOOP line 1&2- loop through each agrunit, and loop through each UNITS that is
*             aggregated to the particular agrunit, and loop through each
*             UNIT that has been aggregated to that UNITS. For each occurance
```

```
*             aggregate that UNIT to the agrunit and remove the aggregation
*             of that UNIT to UNITS. This is saying that if tract a is
*             aggregated to tract b and tract b is aggregated to tract c,
*             then tract a is going to be aggregated to tract c instead of to
*             tract b

LOOP(AGRUNIT,LOOP(UNITS$Aggregate(AGRUNIT,UNITS),LOOP(UNIT$Aggregate(UNITS,UNIT),
   Aggregate(AGRUNIT,UNIT)=1; Aggregate(UNITS,UNIT)=0;)));


**LOOP line 1&2- If a unit is not aggregated to another unit and no unit is
*             aggregated to it, then it is aggregated to itself

Loop(UNIT, if(sum(UNITS,Aggregate(UNITS,UNIT)) EQ 0 and
        sum(UNITS,Aggregate(UNIT,UNITS)) eq 0, Aggregate(UNIT,UNIT)=1;  ));

PARAMETER agrunitpop(UNIT);
       agrunitpop(AGRUNIT)=UNITINFO(AGRUNIT,'POP');

SCALAR totpop population of kentucky by adding up the agrunit populations
     maxagrpop largest population of an agrunit
     minagrpop smallest population of an agrunit
     avgagrpop average population of an agrunit;

       totpop=sum(AGRUNIT, agrunitpop(AGRUNIT));
       maxagrpop=smax(AGRUNIT,agrunitpop(AGRUNIT));
       minagrpop=smin(AGRUNIT,agrunitpop(AGRUNIT));
       avgagrpop=totpop/totaggregate;

*equate aggregate unit populations for testing model validity

parameter CountyMap(UNIT,COUNTIES) 1 if a unit is in a particular county;
       CountyMap(UNIT,COUNTIES)$(UNITINFO(UNIT,'COUNTY') eq Ord(COUNTIES))=1;





*Now aggregate units for county division model all units in a county (except 3
*northern counties) are aggregate to the unit with max population

SET INDIVISIBLE(COUNTIES) counties with less than 90% of mean district pop
      DIVISIBLE(COUNTIES) others (three northern counties);
```

```
    INDIVISIBLE(COUNTIES)$(sum(UNIT$CountyMap(UNIT,COUNTIES),
                ORIGUNITINFO(UNIT,'POP')) le 0.90*AvgDistPOP)=1;
    DIVISIBLE(COUNTIES)=NOT INDIVISIBLE(COUNTIES);

  parameter InDIV(UNIT) units that are in indivisible counties
      MaxPopUnit(COUNTIES) population of unit with largest pop. in that county
      MaxPopUnitId(COUNTIES) ordinal # of unit with largest population in that
county;

      InDIV(UNIT)$Sum(INDIVISIBLE,CountyMap(UNIT,INDIVISIBLE))=1;

  PARAMETER UnitPop(UNIT);
      UnitPop(UNIT)=ORIGUNITINFO(UNIT,'POP');

  **LOOP line 1-   loop through each indivisible county
  **Loop line 2-   maxpopunit=largest population of any unit within that county
  **LOOP line 3&4- maxpopunitid=largest ordinal # of any unit within that county
  *        that has the largest population

  LOOP(INDIVISIBLE,

MaxPopUnit(INDIVISIBLE)=SMAX(UNIT$CountyMap(Unit,INDIVISIBLE),UnitPop(UNIT))
;
     MaxPopUnitId(INDIVISIBLE)=SMAX(UNIT$(CountyMap(UNIT,INDIVISIBLE)
and
           MaxPopUnit(INDIVISIBLE) eq UnitPop(UNIT)),UnitInd(UNIT)););

PARAMETER newagreg(UNIT)the tracts with the largest population there will be one for each
of the indivisible counties;

  **LOOP line 1-   loop through each unit that is in an indivisible county, then
  *        loop through each indivisible county
  **LOOP line 2&3- If the population of that tract = the maxpopulation of that
  *        county and its ordinal # is the same as the ordinal # of the
  *        tract with the largest population, then it is a newagreg

   loop(Unit$InDIV(unit), LOOP(INDIVISIBLE,
      IF(UnitPop(UNIT) = MaxPopUnit(INDIVISIBLE) and
        UnitInd(UNIT) = MaxPopUnitId(INDIVISIBLE), newagreg(UNIT)=1;);));

  *consider all tracts in divisible counties as aggregate units
  AGRUNIT(UNIT)=YES;

  *subtract units in indivisible counties
  AGRUNIT(UNIT)$InDIV(UNIT)=NO;
```

```
SET NEWAGRUNIT(UNIT) they are the tracts with the largest population in each
indivisible county;

NEWAGRUNIT(UNIT)$newagreg(UNIT)=YES;
**AGRUNITS are all the individual tracts in the divisible counties + the largest
*         tracts in each indivisible county
AGRUNIT(UNIT)= AGRUNIT(UNIT)+NEWAGRUNIT(UNIT);

AgrUnitPop(UNIT)=0;
Agrunitpop(AGRUNIT)=UnitPop(AGRUNIT);

*Update aggregate unit pop in indivisible counties as total county population
**LOOP line 1- loop through each indivisible county, then loop through each hub
*            unit in the indivisible counties
**LOOP line 2- the population of that hub unit= sum of the population of all the
*            units in that county
LOOP(INDIVISIBLE,loop(NEWAGRUNIT$CountyMap(NewAgrUnit,indivisible),

Agrunitpop(NEWAGRUNIT)=Sum(Unit$CountyMap(Unit,indivisible),UnitPop(UNIT));));
display Agrunitpop, agrunit;

PARAMETER CountyAggregate(UNITS,UNIT) one when a unit is aggregated to a
units;

**LOOP line 1- Look at every unit that is in an indivisible county, then look at
*            each hub unit of the indivisible counties
**LOOP line 2- If the unit is part of the same county as the hub unit then that
*             unit is aggregated to that mother unit

    Loop(UNIT$INDIV(UNIT),Loop(NEWAGRUNIT,
        If(Unitinfo(UNIT,'County') = Unitinfo(NEWAGRUNIT,'County'),
         CountyAggregate(NEWAGRUNIT,UNIT)=1;)))

SCALAR radius /100/
      MaxDist largest distance from any unit to any center;
* MaxDist is introduced to reduce the model size
MaxDist=smax((CENTER,UNITS), DIST(CENTER,UNITS));

PARAMETER dist_to_center(UNIT);
        dist_to_center(UNIT)=1000;

PARAMETER totalpop(UNIT) population of areas w radius < 0.20 MaxDist ;
        totalpop(CENTER)=sum(AGRUNIT$(Dist(CENTER,AGRUNIT) le
0.20*MaxDist),agrunitpop(AGRUNIT));

VARIABLE MIND  radius of area around centers including 2 times avg district pop
```

```
POSITIVE VARIABLE D radius of min circular area around centers;
BINARY VARIABLE X(UNIT)
EQUATION SUBOBJ
        SETMINPOP
        SETDIST(UNITS);

SUBOBJ.. MIND=e=D;
SETMINPOP..sum(AGRUNIT$(dist_to_center(agrunit) le radius),
            agrunitpop(AGRUNIT)*X(AGRUNIT))=g= 2*AvgDistPop;

SETDIST(AGRUNIT)$(dist_to_center(AGRUNIT) le radius)..
                dist_to_center(AGRUNIT)*X(AGRUNIT)=l=D;

MODEL submodel /SUBOBJ,   SETMINPOP,   SETDIST/;
*option solprint=on;

parameter ordunit(UNIT); ordunit(UNIT)=Ord(UNIT);

**Centerradius is the minimum radius of an area around each center that can make
*           up 2 times the average district population
parameter  CENTERRADIUS(UNIT)

         CENTERRADIUS(UNIT) =0;



**LOOP line 1&2- loop through each center, the radius is the max distantance of
*              an agrunit to that particular centerthat is within 20% of the
*              maximum distance of any unit to a center
**LOOP line 3&4- distance to center looks at every agrunit that is within the
*              radius and records its distance to that particular center, then
*              reset the X variable to 0
* LOOP line 5-   Solve the model for the minimum distance needed around each
*              center to make up 2* average district population
**LOOP line 6- Centerradius=the minimum radius of an area around each center
*            that can make up 2 times the average district population
**LOOP line 7- reset dist_to_ center to 1000

option optcr=0.0;
loop(CENTER,
          RADIUS=SMAX(AGRUNIT$(Dist(CENTER,AGRUNIT) le
0.20*MaxDist),Dist(CENTER,AGRUNIT));
           dist_to_center(agrunit)$(Dist(CENTER,AGRUNIT) le
radius)=Dist(CENTER,AGRUNIT) ;
            X.l(AgrUnit)=0;
            SOLVE  submodel using MIP minimizing MIND;
```

```
                    CENTERRADIUS(CENTER)= MIND.l;
                    dist_to_center(AGRUNIT)=1000;);

        SCALAR tflag total possible allocations that are allowed
            MinCount smallest number of centers that any tract can be attatched to
            MaxCount largest number of centers that any tract can be attatched to
            AvgCount average number of centers that any tract can be attached to;

        PARAMETER  POPDIST(UNIT,UNITS) population times distance used in objfcn
weights
                  FLAG(UNIT,UNITS) flag controlling unit assignment to districts
                  DistCount(UNIT)  number of centers each agrunit can be attached to
                  Accounted(UNIT) check if all agrunits are accounted for within the flagged
circular areas around each center
                  UnAccounted(UNIT) unaccounted tracts ;

          POPDIST(CENTER,AGRUNIT)=Dist(CENTER,AGRUNIT)*
agrunitpop(AGRUNIT)/1000;
        *If a unit is within the centerradius, then it can possibly be attached to that center
           FLAG(CENTER,UNIT)$(Dist(CENTER,UNIT) le CENTERRADIUS(center))=1;

        **If that unit can be attached to atleast 1 center then it is accounted for
          Accounted(UNIT)$Sum(CENTER,FLAG(CENTER,UNIT))=1;
          UnAccounted(UNIT)=1-Accounted(UNIT);

          DistCount(AGRUNIT)=Sum(CENTER,FLAG(CENTER,AGRUNIT));
          tflag=sum((center,AGRUNIT),FLAG(CENTER,AGRUNIT));
          MinCount=Smin(AGRUNIT,DistCount(AGRUNIT));
          MaxCount=Smax(AGRUNIT,DistCount(AGRUNIT));
          AvgCount=tflag/Sum(AGRUNIT,1);

        SCALAR   agrunitid aggregate unit id
              attached number of centers each unit can be attached to;
        loop(agrunit, agrunitid=ordunit(agrunit);
                 attached=sum(center,FLAG(CENTER,AGRUNIT));
           if(attached eq 1, display 'this unit can be attached to one center', agrunitid;););
        loop(agrunit, agrunitid=ordunit(agrunit);
                 attached=sum(center,FLAG(CENTER,AGRUNIT));
           if(attached eq 2, display 'this unit can be attached to two centers', agrunitid;););


        PARAMETER  distpop(UNIT) district populations
                  devperc(UNIT) percent deviation from mean pop
                  DISTRICTLEV(UNITS,UNIT) report solution
                  RADIUSfinal(UNITS) final radius of districts  ;
```

```
****Solve the county division model

VARIABLE TOTALPOPDIST population weighted total distance to be minimized

*first argument of DISTRICT is center id second argument is assigned unit id
BINARY VARIABLE DISTRICT(UNIT,UNITS) determines assignment of unit to
districts

PARAMETER  CCDIST(COUNTIES,UNIT) min distance between counties and centers
      CCFLAG(COUNTIES,UNIT) flag for county-center pairs;


CCDIST(COUNTIES,CENTER)=SMIN(UNIT$CountyMap(Unit,Counties),DIST(UNIT,CENT
ER));
 CCDIST(COUNTIES,UNIT)=0;
 CCFLAG(COUNTIES,CENTER)$(CCDIST(COUNTIES,CENTER) le
CENTERRADIUS(CENTER))=1;


SCALAR penalty penalty for having a county division /5000/;

BINARY VARIABLE   PARTITION(COUNTIES,UNIT)

EQUATIONS
   NUMDISTRICT    fixes the total number of districts
   CENTERDISTRICT forces each unit to belong to a district
   CREATE        Unit assignment can occur if a district is created @ CENTER
   POPUPPER      Upper limit for district populations
   POPLOWER      Lower limit for district populations
   OBJ2          objective equation with county partitioning
   COUNTYDIV     determines if a county is divided between districts
   AUXCOUNTYDIV  auxiliary county division constraint  ;


NUMDISTRICT.. SUM(CENTER,DISTRICT(CENTER,CENTER))=E=TotalDistrict;

CENTERDISTRICT(AGRUNIT).. SUM(CENTER$Flag(CENTER,AGRUNIT),
                     DISTRICT(CENTER,AGRUNIT))=E=1;

CREATE(CENTER)..  SUM(AGRUNIT$Flag(CENTER,AGRUNIT),
DISTRICT(CENTER,AGRUNIT))
                 =L= DISTRICT(CENTER,CENTER)*Group;
```

```
POPUPPER(CENTER).. SUM(AGRUNIT$Flag(CENTER,AGRUNIT),
            agrunitpop(AGRUNIT)*DISTRICT(CENTER,AGRUNIT))
               =L= (1+Limit)*AvgDistPop*DISTRICT(CENTER,CENTER);

POPLOWER(CENTER).. SUM(AGRUNIT$Flag(CENTER,AGRUNIT),
            agrunitpop(AGRUNIT)*DISTRICT(CENTER,AGRUNIT))
               =G=(1-Limit)*AvgDistPop*DISTRICT(CENTER,CENTER);



OBJ2..TOTALPOPDIST =E= SUM((CENTER,AGRUNIT)$Flag(CENTER,AGRUNIT),
                  PopDist(CENTER,AGRUNIT)*DISTRICT(CENTER,AGRUNIT)) +
              Sum((DIVISIBLE,CENTER)$CCFLAG(DIVISIBLE,CENTER),
                   penalty*PARTITION(DIVISIBLE,CENTER));

COUNTYDIV(DIVISIBLE,CENTER)$CCFLAG(DIVISIBLE,CENTER)..
     Sum(AGRUNIT$(CountyMap(AgrUnit,DIVISIBLE)*Flag(CENTER,AGRUNIT)),
       DISTRICT(CENTER,AGRUNIT)) =l=
Group*PARTITION(DIVISIBLE,CENTER);

AUXCOUNTYDIV(CENTER)..   Sum(DIVISIBLE$CCFLAG(DIVISIBLE,CENTER),
      PARTITION(DIVISIBLE,CENTER)) =l=group*DISTRICT(CENTER,CENTER) ;


MODEL COUNTYDIVISION /OBJ2,NUMDISTRICT,CENTERDISTRICT,CREATE,
              POPUPPER,POPLOWER,COUNTYDIV,AUXCOUNTYDIV/    ;

option optcr=0.0;
option reslim=50000
SOLVE COUNTYDIVISION USING MIP MINIMIZING TOTALPOPDIST    ;

 distpop(center)=
SUM(AGRUNIT,agrunitpop(AGRUNIT)*DISTRICT.l(CENTER,AGRUNIT));
  devperc(center)$distpop(center)=abs(distpop(center)- avgdistpop) /avgdistpop;

  DISTRICT.l(CENTER,UNIT)$INDIV(UNIT)=
     sum(AGRUNIT$CountyAggregate(AGRUNIT,UNIT),
DISTRICT.l(CENTER,AGRUNIT)) ;
  DISTRICTLEV(UNIT,CENTER)$distpop(center) =DISTRICT.l(CENTER,UNIT);
  RADIUSfinal(CENTER)$DISTRICT.l(CENTER,CENTER)=
       SMAX(UNIT$DISTRICT.l(CENTER,UNIT),ORIGDIST(CENTER,UNIT));

PARAMETER DIVISION(Counties) number of times counties are divided;
  Division(Counties)=sum(CENTER$DISTRICTLEV(CENTER,CENTER),
            PARTITION.l(COUNTIES,CENTER));
```

References

Altman, M. (1995). The Consistency and Effectiveness of Mandatory District Compactness Rules. In *Annual Meeting of the MidWestern Political Science Association, Chicago*.

Bação, F., Lobo, V., & Painho, M. (2005). Applying genetic algorithms to zone design. *Soft Computing-A Fusion of Foundations, Methodologies and Applications*, *9*(5), 341-348.

Backer v. Carr. (1962). *U.S.* (Vol. 369, p. 186).

Bartlett v. Strickland. (2009). *U.S.* (Vol. 556).

Cerdeira, J. Orestes, K. J. Gaston, and L. S. Pinto (2005). Connectivity in priority area selection for conservation. *Environmental Modeling and Assessment*, 10(2), 183–192

Chou, C. I., & Li, S. P. (2006). Taming the Gerrymander--Statistical Physics Approach to Political Districting Problem. *Physica A: Statistical Mechanics and its Applications*, *369*(2), 799-808.

Colegrove v. Green. (1946). *U.S.* (Vol. 328, p. 549).

Courtney, J. C. (2004). Redistricting: What the United States Can Learn from Canada. *Election Law Journal*, *3*(3), 488-500.

Fischer v. State Board of Elections. (1994). *S.W.2d* (Vol. 879, p. 475).

Forrest, E. (1964). Apportionment by computer. *American behavioral scientist*, *8*(4), 23.

Galvão, L. C., Novaes, A. G. N., Souza de Cursi, J. E., & Souza, J. C. (2006). A multiplicatively-weighted Voronoi diagram approach to logistics districting. *Computers & Operations Research*, *33*(1), 93-114.

Garfinkel, R. S., & Nemhauser, G. L. (1970). Optimal political districting by implicit enumeration techniques. *Management Science*, *16*(8), 495-508.

Harris Jr, C. C. (1964). A scientific method of districting. *Behavioral Science*, *9*(3), 219-225.

Kentucky Legislative Research Commission. (n.d.). Maps. *Kentucky Legislature*. Retrieved April 19, 2010, from http://www.lrc.ky.gov/gis/maps.htm

Mehrotra, A., Johnson, E. L., & Nemhauser, G. L. (1998). An optimization based heuristic for political districting. *Management Science*, 44(8), 1100-1114.

Nagel, S. S. (1965). Simplified bipartisan computer redistricting. *Stanford Law Review*, *17*(5), 863-899.

Önal, H. and R. Briers (2006). Optimum Selection of a Connected Conservation Reserve Network. *Operations Research*, 54(2), 379-388.

Reynolds v. Sims. (1964). *U.S.* (Vol. 377, p. 533).

Spann, A., Kane, D., & Gulotta, D. (2007). Electoral Redistricting with Moment of Inertia and Diminishing Halves Models. *UMAPJournal*, 281.

Testimony Presented at Town Hall Meetings. (2001, July 10). . Kansas Legislative Research Department. Retrieved from http://skyways.lib.ks.us/ksleg/KLRD/Redistrct/documents/testimony_presented_at_town_hall_mtgs.pdf

Thoreson, J. D., & Liittschwager, J. M. (1967). Computers in behavioral science. Legislative districting by computer simulation. *Behavioral Science*, *12*(3), 237-247.

Tobler, W. (2004). Thirty five years of computer cartograms. *Annals of the Association of American Geographers*, *94*(1), 58-73.

Tobler, W. R. (1973). A continuous transformation useful for districting. *Annals of the New York Academy of Sciences*, *219*(9), 215-220.

Vickrey, W. (1961). On the prevention of gerrymandering. *Political Science Quarterly*, *76*(1), 105-110.

Weaver, J. B., & Hess, S. W. (1963). A procedure for nonpartisan districting: development of computer techniques. *Yale Law Journal*, *73*(2), 288-308.

Weisstein, E. (n.d.). Convex Polygon -- from Wolfram MathWorld. Retrieved April 12, 2010, from http://mathworld.wolfram.com/ConvexPolygon.html

Weisstein, E. (n.d.). Great Circle. *Mathworld-A Wolfram Web Resource*. Retrieved April 12, 2010, from http://mathworld.wolfram.com/GreatCircle.html

Wesberry v. Sanders. (1964). *U.S.* (Vol. 376, p. 1).

Williams Jr, J. C. (2005). Political redistricting: a review. *Papers in Regional Science*, *74*(1), 13-40.

Williams J. C., ReVelle C. S., Levin S. A. 2005. Spatial attributes and reserve design models: A review. Environmental Modeling Assessment, 10(3), 163-181.

Young, H. P. (1988). Measuring the compactness of legislative districts. *Legislative Studies Quarterly*, *13*(1), 105-115.